AGT対応

- 入門から最近の進展まで -

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duality (双対性)

Theory A

||

Theory B

duality (双对性): 例1

Theory A

$$\mathcal{L}_A = \bar{\psi} i(\sigma^\mu \partial_\mu + m)\psi - \frac{g}{2}(\bar{\psi}\sigma^\mu\psi)^2$$

2d massive Thirring model

Theory B

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu \phi)^2 - h\cos\phi$$

2d sine-Gordon model

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duality (双对性): 例2 Montonen-Olive S-duality

Theory A

$$\mathcal{L}_A = \frac{1}{4(e_A)^2}\mathrm{Tr}(F_{\mu\nu})^2$$

4d YM

Theory B

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4d YM

$$(e_A)^2 = \frac{1}{(e_B)^2}$$

Theory B

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4d YM

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4d YM

1. Sketch of AGT correspondence

AGT: a kind of duality

Theory A

4d N=2 susy gauge theory

Theory B

2d conformal field theory (CFT)

AGT: a kind of duality

Theory A

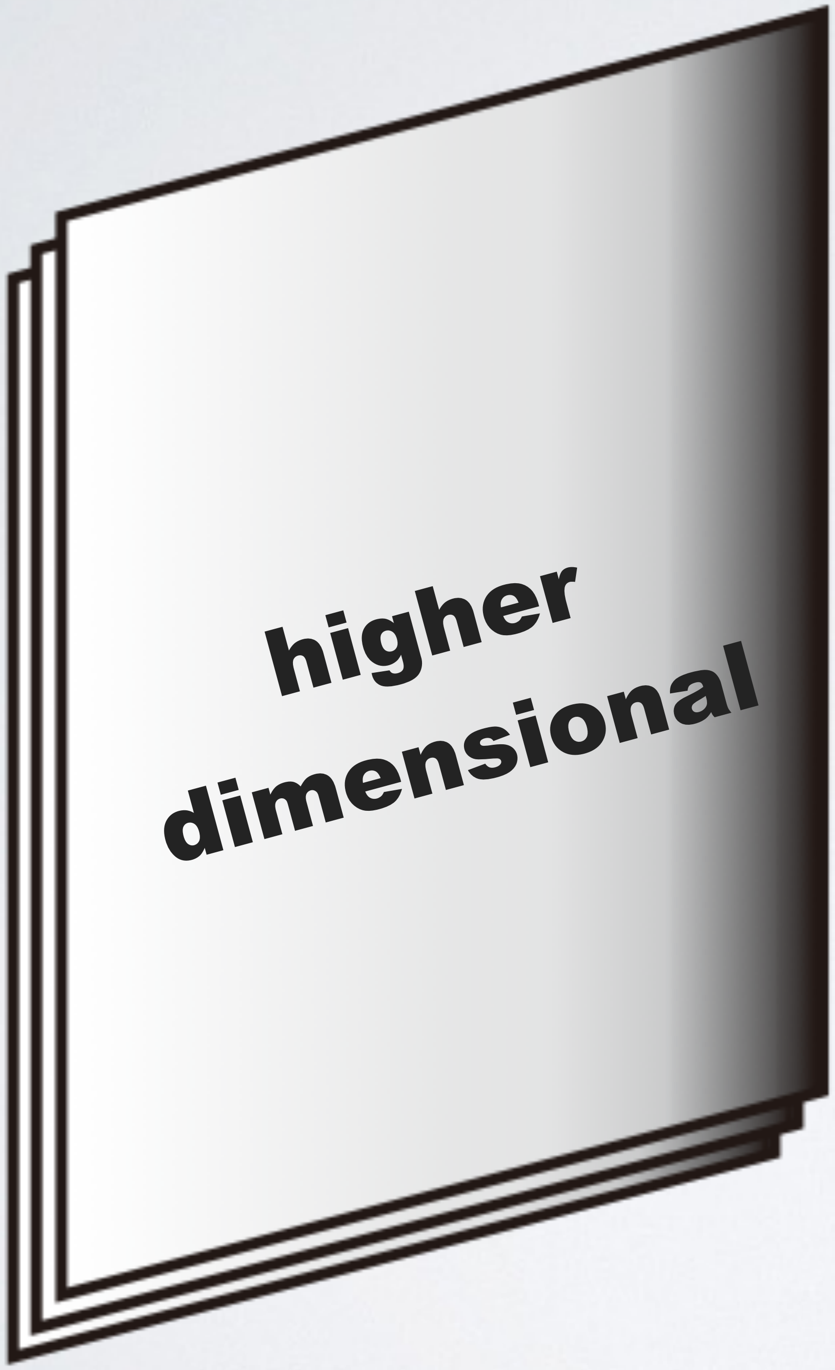
4d N=2 susy gauge theory

10d string theory

Theory B

2d conformal field theory (CFT)

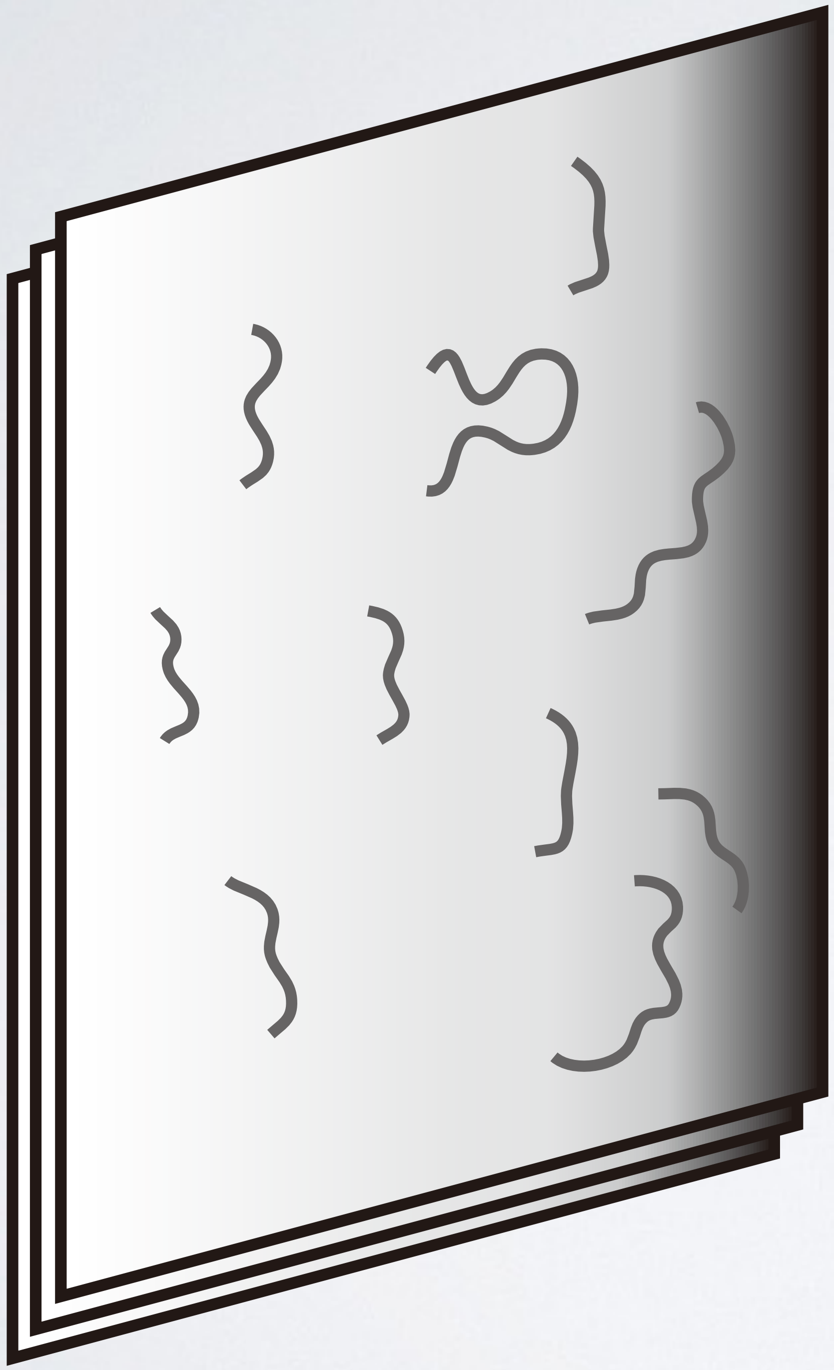
D-branes



hyper-plane in 10d spacetime

Open strings have their ends on it

Open string and gauge fields



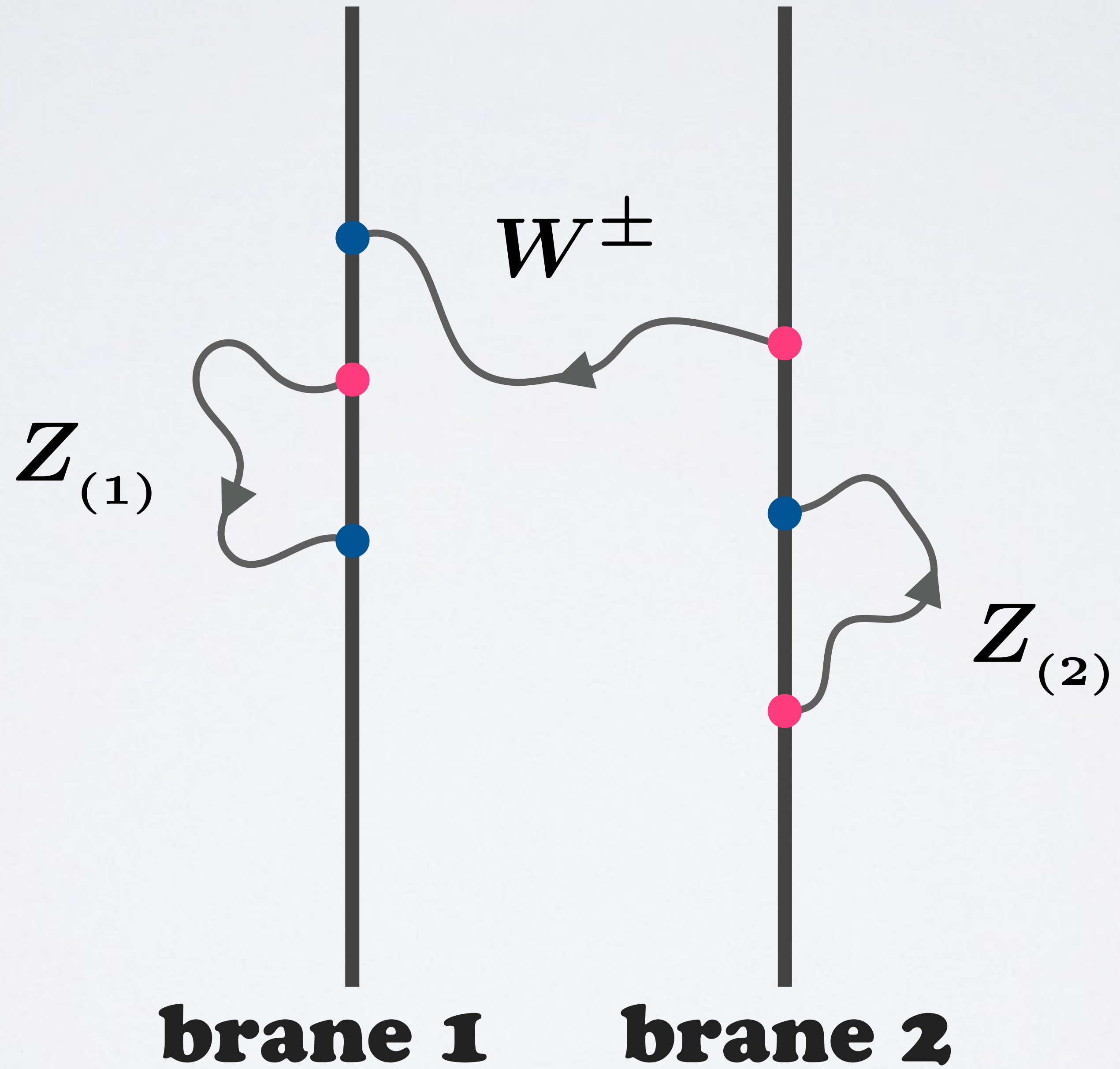
hyper-plane in 10d spacetime

Open strings have their ends on it

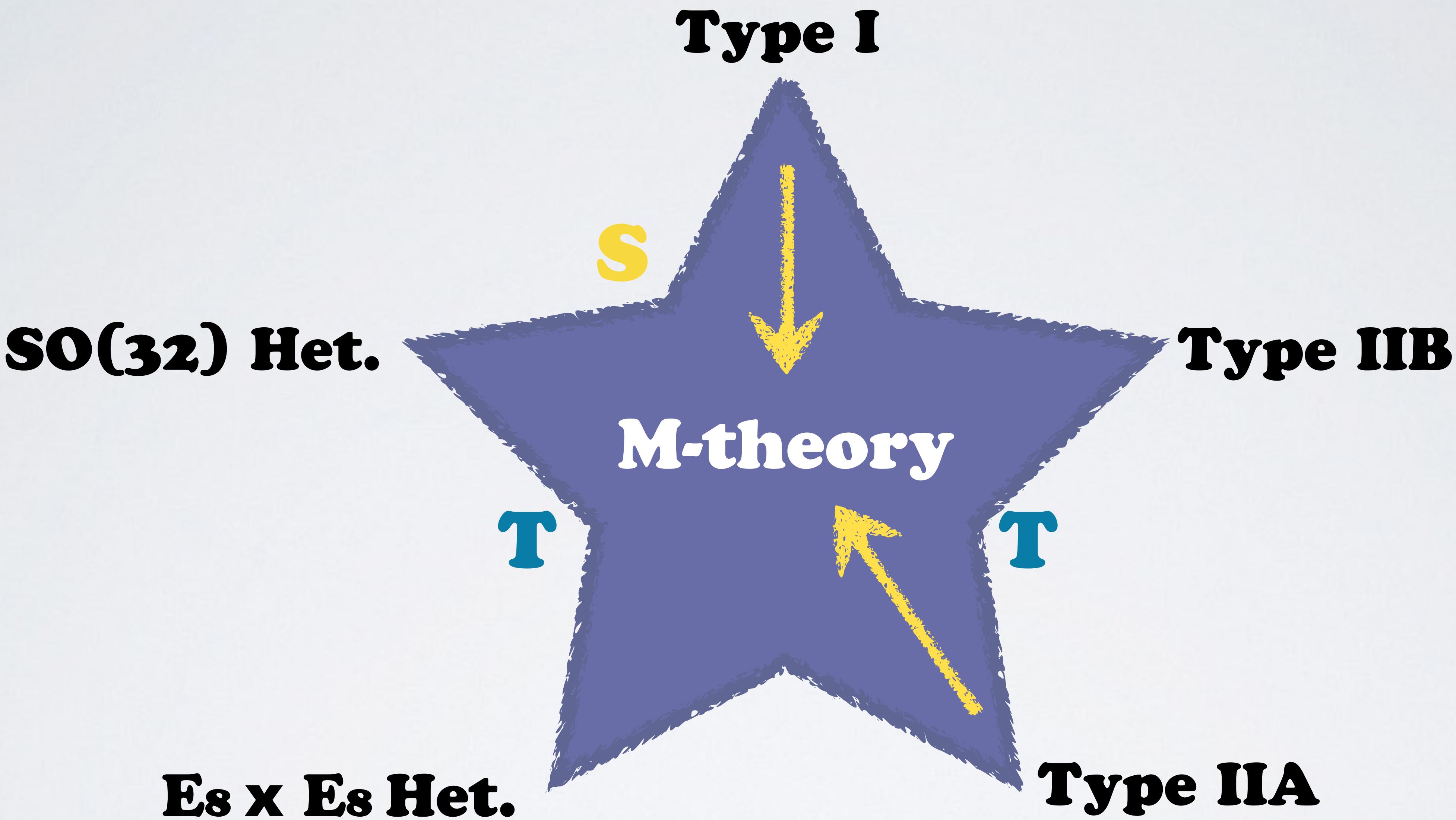
→ SU(N)

Open string and gauge fields

$U(2) \rightarrow U(1)_1 \times U(1)_2$



Perturbative String Theories



AGT via M-theory

M5 on Cylinder → 4D Gauge Theory

$\mathbb{R}^4 \times$ 

N_c **M5s** → $SU(N_c)$

AGT via M-theory

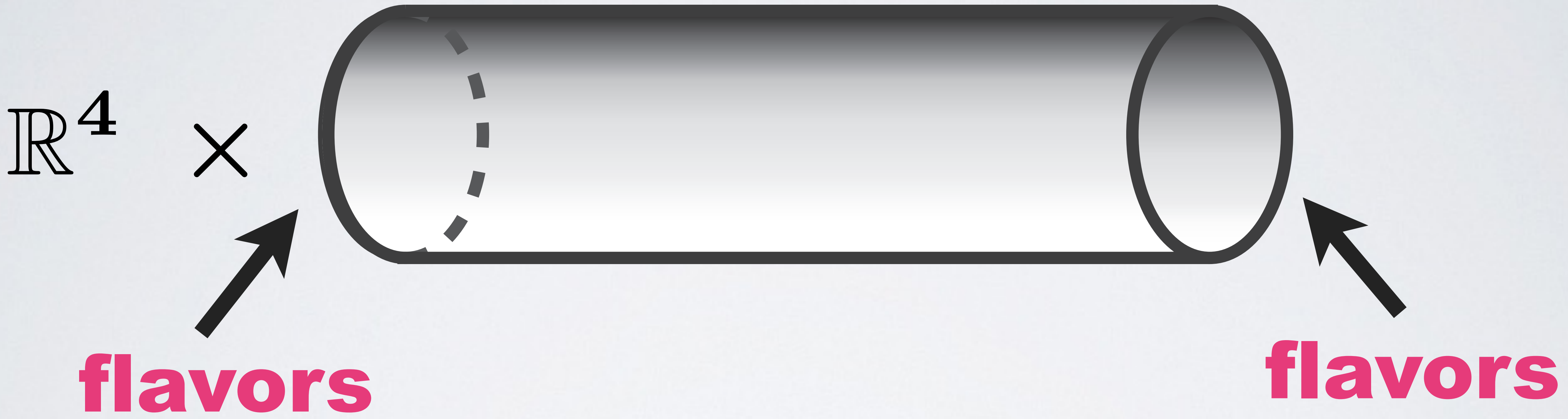
M5 on Cylinder → **4D Gauge Theory**

$\mathbb{R}^4 \times$ 

N_c **M5s** → $SU(N_c)$

Quarks ?

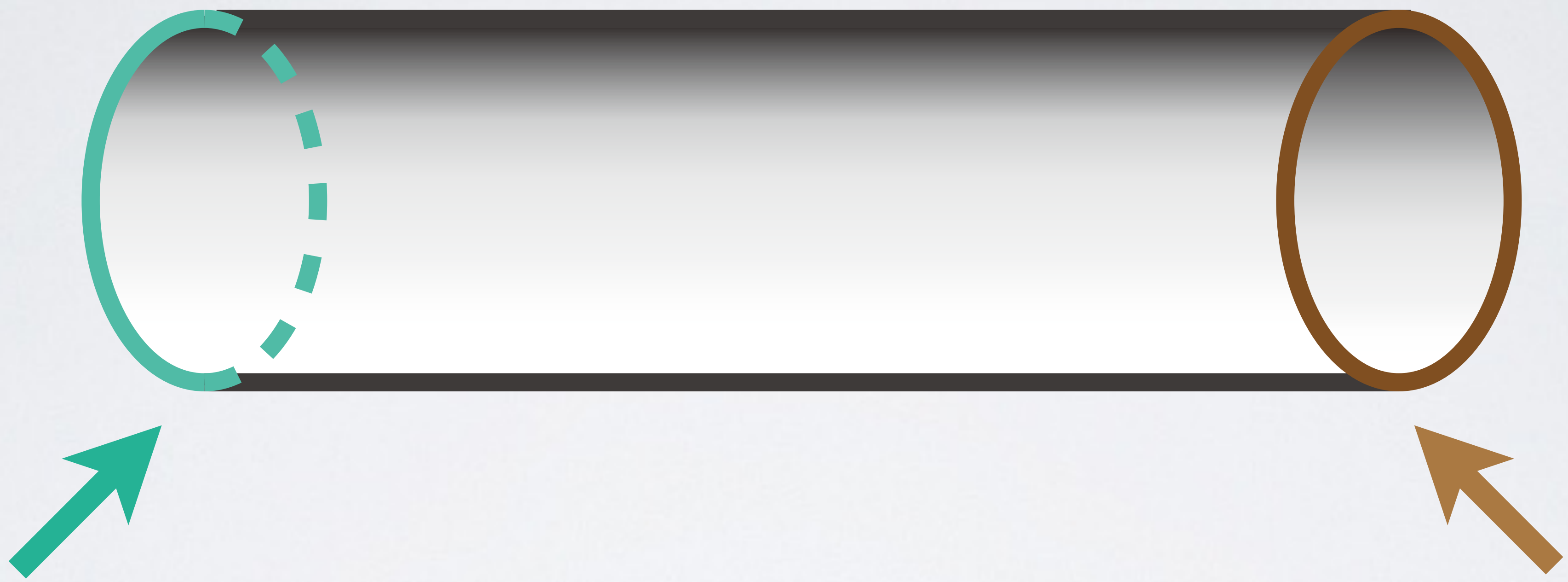
AGT via M-theory



→ **flavors live on the edges**

AGT via M-theory

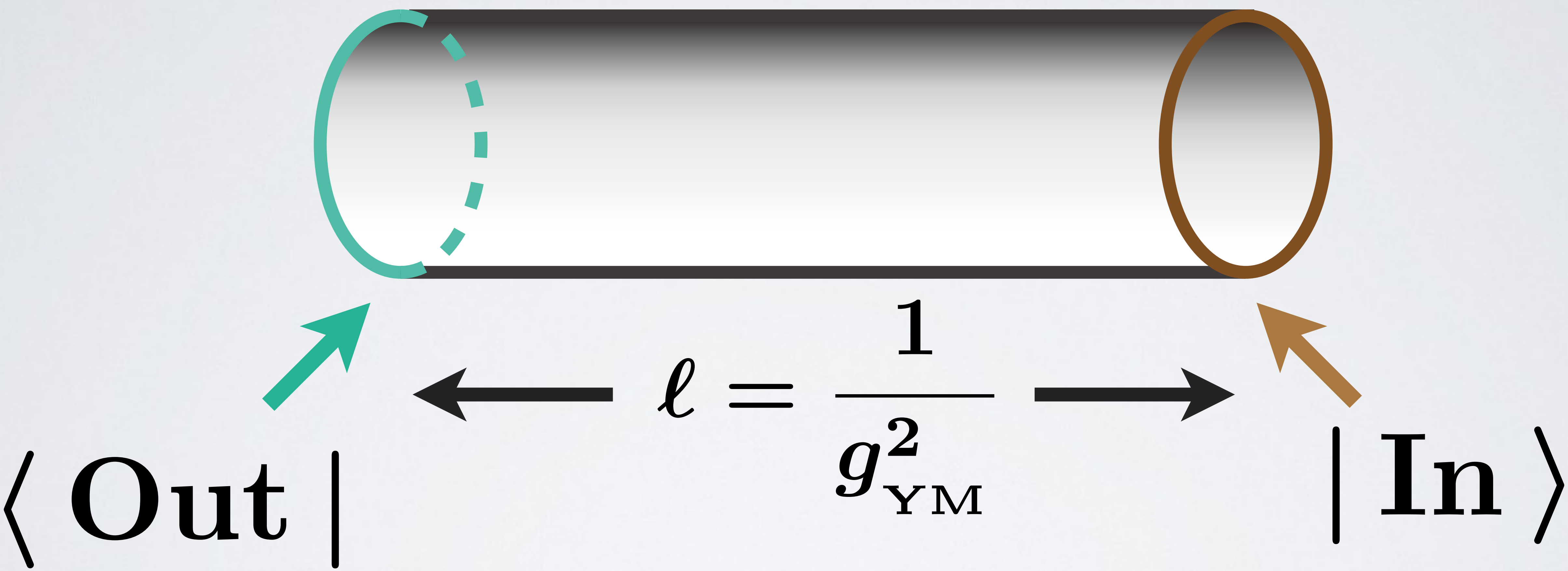
Boundary Condition as a State



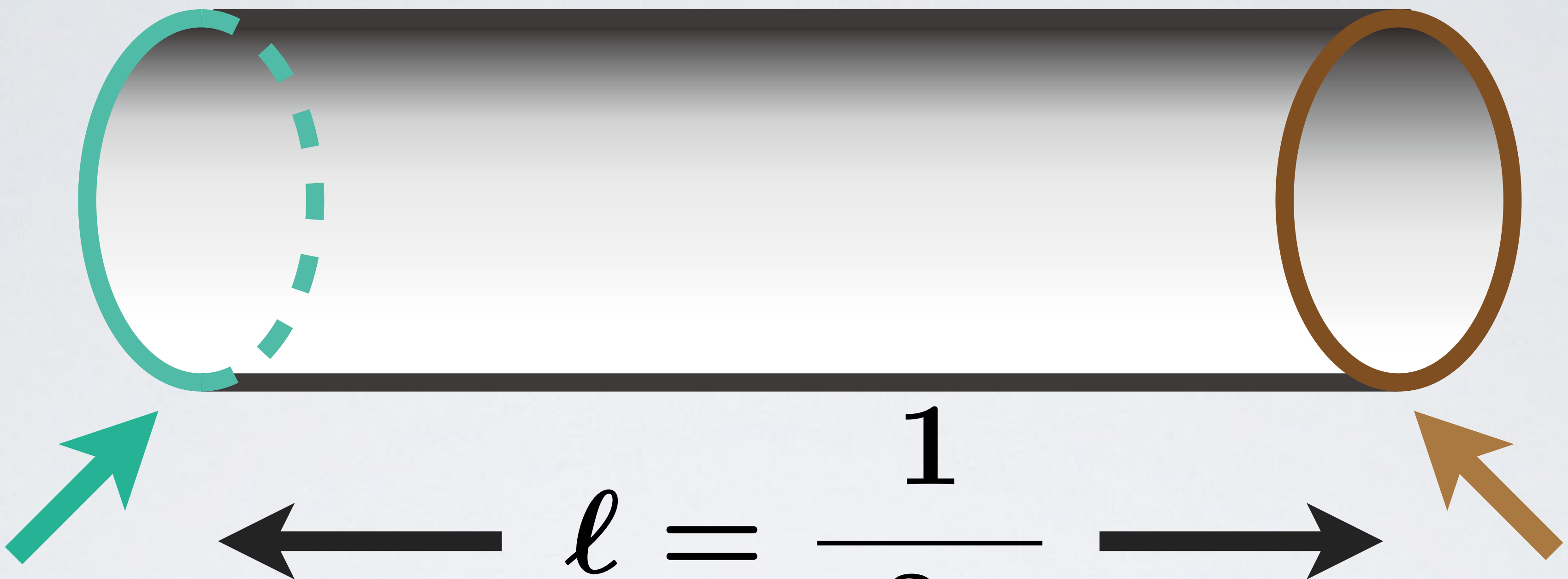
$$\langle \, \mathbf{Out} \, | \qquad\qquad | \, \mathbf{In} \, \rangle$$

AGT via M-theory

Gauge Coupling is the Length



AGT via M-theory



$\langle\, \mathbf{Out}\, |$ $\qquad \ell = \dfrac{1}{g^2_{\mathrm{YM}}}$ $\qquad |\, \mathbf{In}\, \rangle$

$$Z_{4D} = \langle\, \mathbf{Out}\, | \Lambda^{2N_c L_0} |\, \mathbf{In}\, \rangle$$

Partition function is Matrix Element

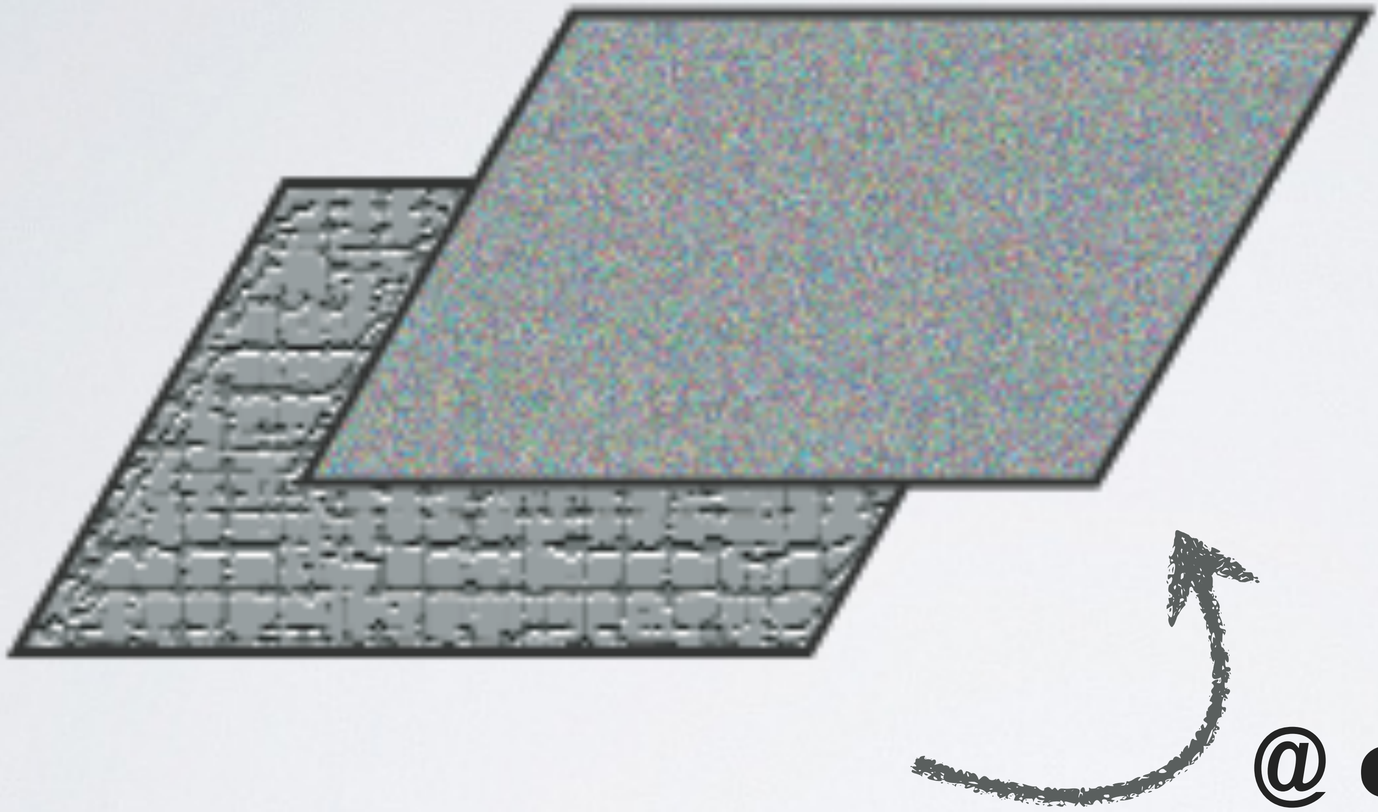
AGT via M-theory

What's the state?

state in CFT with Virasoro symmetry

$$[l_m, l_n] = (m - n) l_{m+n}$$

2d CFT describes critical phenomena



@ critical point

No typical scale

conformal symmetry

$$z \to f(z) \simeq z + \epsilon(z)$$

conformal symmetry

$$z \to f(z) \simeq z + \epsilon(z)$$

$$= z + \sum_n \epsilon_n \left(z^{n+1} \frac{\partial}{\partial z} \right) z$$

conformal symmetry

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$$= z + \sum_n \epsilon_n \left(z^{n+1} \frac{\partial}{\partial z} \right) z$$

conformal symmetry

$$l_n = -z^{n+1}\frac{\partial}{\partial z}$$

$$[l_m, l_n] = (m-n)l_{m+n}$$

2. instanton & partition function

SU(2) instanton

minimum of Euclidean Yang-Mills action

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

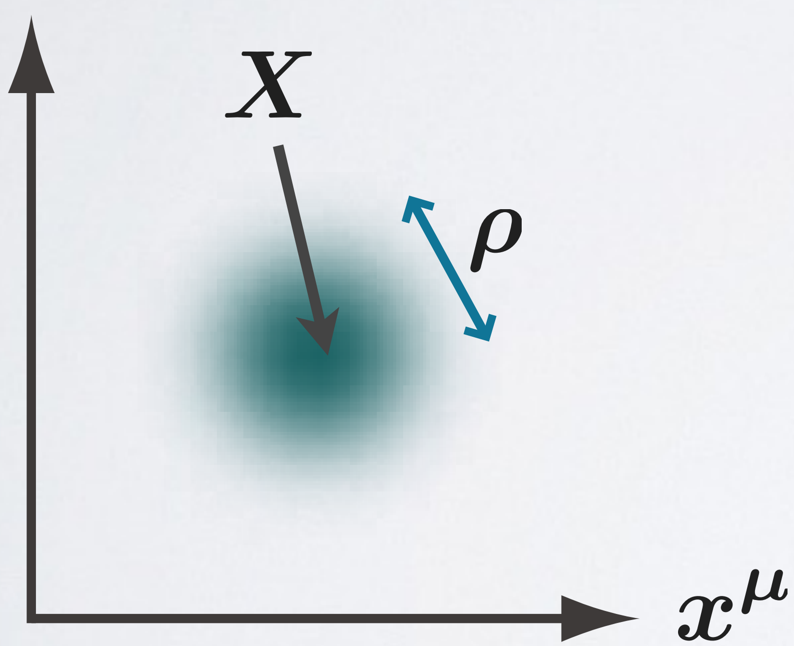
$$S = \int d^4x (F_{\mu\nu})^2$$

SU(2) instanton

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda} \quad \rightarrow \quad A^a_\mu(x) = \sum_{b=1}^{3} M^{ab}\frac{(x-X)^\nu \eta^b_{\mu\nu}}{(x-X)^2+\rho^2}$$

self-dual equation

1-instanton solution



- X^μ — center of the instanton
- ρ — size
- $M \in SO(3)$

$\mathcal{M}_k$ **: k-instanton parameter space**

$$\mathcal{M}_1 = \mathbb{R}^4 \times \mathbb{R}_+ \times SO(3)$$

instanton partition function

$$Z^{\mathrm{Nek}} = \sum_{k=0} \Lambda^{4k} \int_{\mathcal{M}_k} dV$$

instanton partition function

$$Z^{\mathrm{Nek}} = \sum_{k=0} \Lambda^{4k} \int_{\mathcal{M}_k} dV$$

$$\int_{\mathbb{R}^2} dX^1 dX^2 \rightarrow \int_{\mathbb{R}^2} dX^1 dX^2 e^{-\epsilon_1((X^1)^2+(X^2)^2)}$$

$$= \frac{1}{\epsilon_1}$$

three $U(1)$**'s :** $\epsilon_1, \ \epsilon_2, \ a$

instanton partition function

1-instanton

2-instanton

$$Z^{\text{Nek}} = 1 + \frac{2\Lambda^4}{\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+\epsilon_2-2a)}$$

$$+\frac{\Lambda^8(8(\epsilon_1+\epsilon_2)+\epsilon_1\epsilon_2-8a^2)}{\epsilon_1^2\epsilon_2^2((\epsilon_1+\epsilon_2)^2-4a^2)((2\epsilon_1+\epsilon_2)^2-4a^2)((\epsilon_1+2\epsilon_2)^2-4a^2)}$$

$$+\cdots$$

Origin of instanton partition function

N=2 SUSY

$$Q^a_\alpha,\ \bar{Q}^a_{\dot{\alpha}}$$

$$SO(4) = SU(2)_L \times SU(2)_R$$

Origin of instanton partition function

N=2 SUSY

$$Q^a_\alpha,\ \bar{Q}^a_{\dot{\alpha}}$$

$$SO(4) = SU(2)_L \times SU(2)_R$$

$$Q^1_\alpha \overset{SU(2)_I}{\leftrightarrow} Q^2_\alpha$$

Origin of instanton partition function

twisting N=2 SUSY

$$SO(4) = SU(2)_L \times SU(2)_R$$

$$SU(2)_R = SU(2)_I$$

Origin of instanton partition function

twisting N=2 SUSY

$$SO(4) = SU(2)_L \times SU(2)_R$$

$$\| $$

$$SU(2)_I$$

$$Q_\alpha^\beta \longrightarrow Q \oplus Q_\mu$$

$$(\quad 2 \otimes 2 \longrightarrow 1 \oplus 3 \quad)$$

Origin of instanton partition function

twisting N=2 SUSY

$$Z = \int \mathcal{D}\Phi e^{-\{Q,V\} - \frac{c}{g^2}\int F_{\mu\nu} * F^{\mu\nu} d^4x}$$

$$= \sum_k \sum_{k-instantons} e^{-\frac{c}{g^2}k} Z_{1-loop}$$

3. conformal symmetry

Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

Rep. of SU(2)

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$J_\pm = J_1 \pm i J_2$$

Rep. of SU(2)

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$$J_{\pm} = J_1 \pm i J_2$$

→

$$[J_3, J_{\pm}] = \pm J_{\pm} \qquad [J_+, J_-] = 2J_3$$

Rep. of SU(2)

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→ $$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3$$

→ $$J_3 J_\pm |*\rangle = J_\pm (J_3 \pm 1)|*\rangle$$

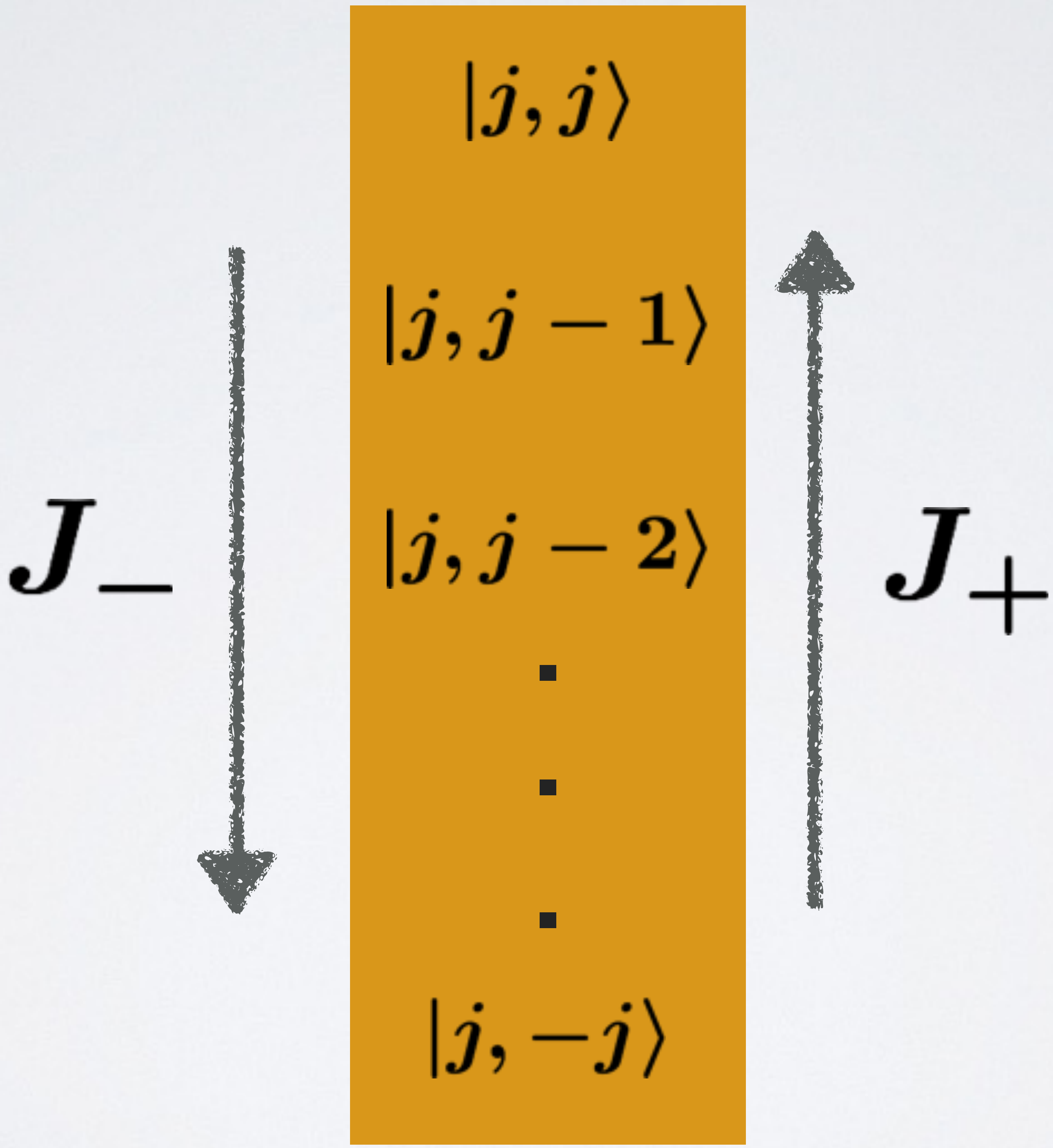
weight shift

Rep. of SU(2)

$\vec{J}^2$ $\quad$ J_3

$$|j, j\rangle$$

Rep. of SU(2)



Rep. of Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_{n>0}|\Delta\rangle = 0$$

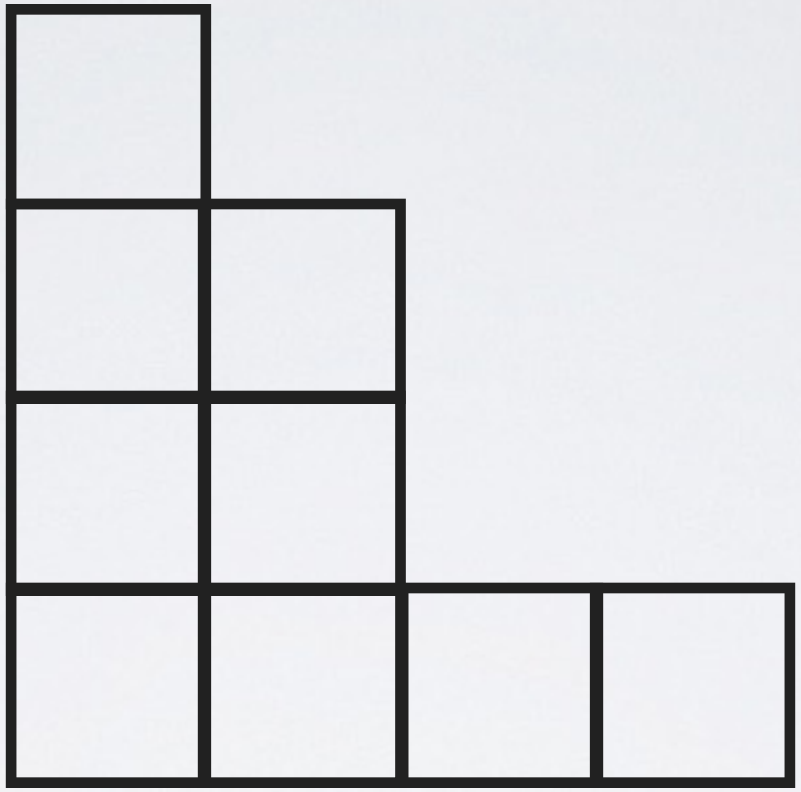
analogy of $|j, j\rangle$

$$L_{-n}L_{-m}\cdots|\Delta\rangle$$

weight	basis
Δ	$\|\Delta\rangle$
$\Delta+1$	$L_{-1}\|\Delta\rangle$
$\Delta+2$	$L_{-2}\|\Delta\rangle$ $\quad L_{-1}{}^2\|\Delta\rangle$
$\Delta+3$	$L_{-3}\|\Delta\rangle$ $\quad L_{-2}L_{-1}\|\Delta\rangle$ $\quad L_{-1}{}^3\|\Delta\rangle$
	$\cdots$
$\Delta+n$	$L_{-Y}\|\Delta\rangle \equiv L_{-Y_1}L_{-Y_2}\cdots\|\Delta\rangle$ $\quad n=\|Y\|=\sum Y_i$

L_{+n}

Young diagram



$$Y = [Y_1, Y_2, Y_3, \ldots] = [4, 3, 1, 1, 0, 0, \ldots]$$

$$L_{-n}L_{-m}\cdots|\Delta\rangle$$

weight	basis
Δ	$\|\Delta\rangle$
$\Delta+1$	$L_{-1}\|\Delta\rangle$
$\Delta+2$	$L_{-2}\|\Delta\rangle \quad {L_{-1}}^2\|\Delta\rangle$
$\Delta+3$	$L_{-3}\|\Delta\rangle \quad L_{-2}L_{-1}\|\Delta\rangle \quad {L_{-1}}^3\|\Delta\rangle$
	$\cdots$
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A building block of AGT relation

Gaiotto state

$$L_1|\Lambda, \Delta\rangle = \Lambda^2|\Lambda, \Delta\rangle \qquad L_{n>1}|\Lambda, \Delta\rangle = 0$$

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ansatz

$$|\Delta, \Lambda\rangle = |\Delta\rangle + c_1 L_{-1}|\Delta\rangle + c_{11} {L_{-1}}^2|\Delta\rangle + c_2 L_{-2}|\Delta\rangle + \cdots$$

Gaiotto state

$$L_1|\Lambda,\Delta\rangle = \Lambda^2|\Lambda,\Delta\rangle \qquad L_{n>1}|\Lambda,\Delta\rangle = 0$$

ansatz

$$|\Delta,\Lambda\rangle = |\Delta\rangle + c_1 L_{-1}|\Delta\rangle + c_{11} {L_{-1}}^2|\Delta\rangle + c_2 L_{-2}|\Delta\rangle + \cdots$$

$$\rightarrow \quad |\Delta,\Lambda\rangle = |\Delta\rangle + \frac{\Lambda^2}{2\Delta} L_{-1}|\Delta\rangle + \cdots$$

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$$\rightarrow \quad |\Delta,\Lambda\rangle = |\Delta\rangle + \frac{\Lambda^2}{2\Delta} L_{-1}|\Delta\rangle + \cdots$$

$$\langle\Delta,\Lambda|\Delta,\Lambda\rangle = 1 + \frac{\Lambda^4}{2\Delta} + \frac{\Lambda^8(c+8\Delta)}{4\Delta(16\Delta^2 - 10\Delta + c(1+2\Delta))} + \cdots$$

An AGT relation for pure SU(2) YM

* $$\langle \Delta, \Lambda' | \Delta, \Lambda' \rangle = 1 + \frac{\Lambda'^4}{2\Delta} + \frac{\Lambda'^8(c+8\Delta)}{4\Delta(16\Delta^2 - 10\Delta + c(1+2\Delta))} + \cdots$$

* $$Z^{\mathrm{Nek}} = 1 + \frac{2\Lambda^4}{\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+\epsilon_2-2a)}$$

$$+ \frac{\Lambda^8(8(\epsilon_1+\epsilon_2)+\epsilon_1\epsilon_2-8a^2)}{\epsilon_1^2\epsilon_2^2((\epsilon_1+\epsilon_2)^2-4a^2)((2\epsilon_1+\epsilon_2)^2-4a^2)((\epsilon_1+2\epsilon_2)^2-4a^2)} + \cdots$$

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AGT dictionary (universal)

$$\epsilon_1\epsilon_2\Lambda'^2 = \Lambda^2 \qquad c = 1 + 6\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} \qquad \Delta = \frac{(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1\epsilon_2} - \frac{a^2}{\epsilon_1\epsilon_2}$$

AGT relation

$$\langle \Lambda, \Delta | \Lambda, \Delta \rangle = Z^{\mathrm{Nek}}$$

4. flavorful states

beta function of N=2 gauge theories

$$\beta_g \propto 2N_c - N_f$$

many of them are asymptotically non-free

generic partition function for SU(N) theory

[Nekrasov, '02]

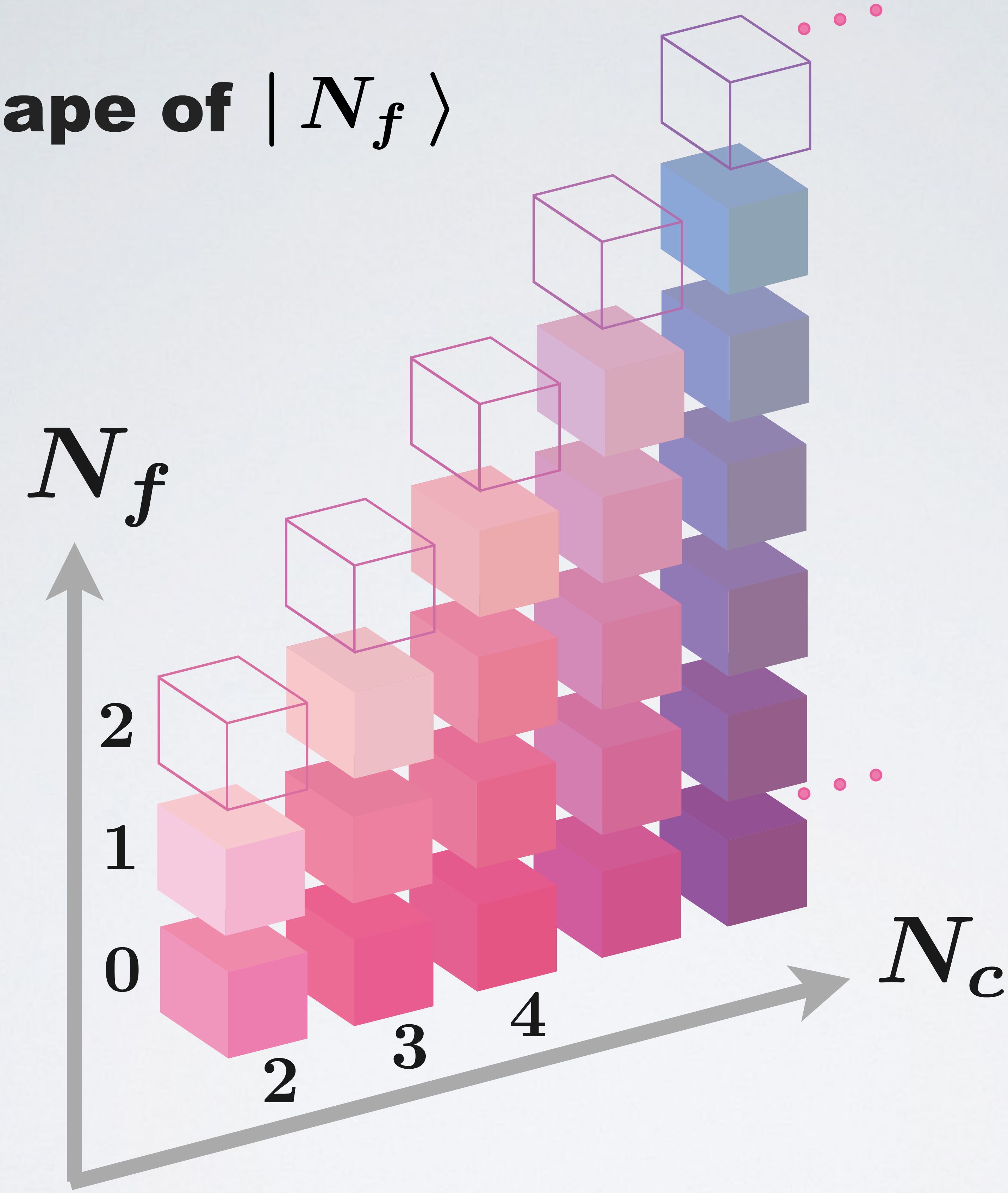
$$Z = \sum_{Y_1, \cdots, Y_N} q^{|\vec{Y}|} z_{\rm vector}(\vec{a}, \epsilon_{1,2}; \vec{Y}) z_{\rm matters}(\vec{a}, m, \epsilon_{1,2}; \vec{Y})$$

$$\begin{aligned} & z_{\rm vector}(\vec{a}, \epsilon_{1,2}; \vec{Y}) \\ & = \prod_{a,b=1}^{N} \prod_{(i,j)\in Y_a} (a_a - a_b - \epsilon_1(Y_{bj}^T - i + 1) + \epsilon_2(Y_{ai} - j + 1))^{-1} \\ & \qquad \times \prod_{(i,j)\in Y_b} (a_a - a_b + \epsilon_1(Y_{bi} - j + 1) - \epsilon_2(Y_{aj}^T - i + 1))^{-1} \end{aligned}$$

AGT has a wide range of generalizations

$$\langle\, N_f - n \,|\, n \,\rangle = Z^{N_f}_{Nek}$$

Landscape of $|\, N_f \,\rangle$



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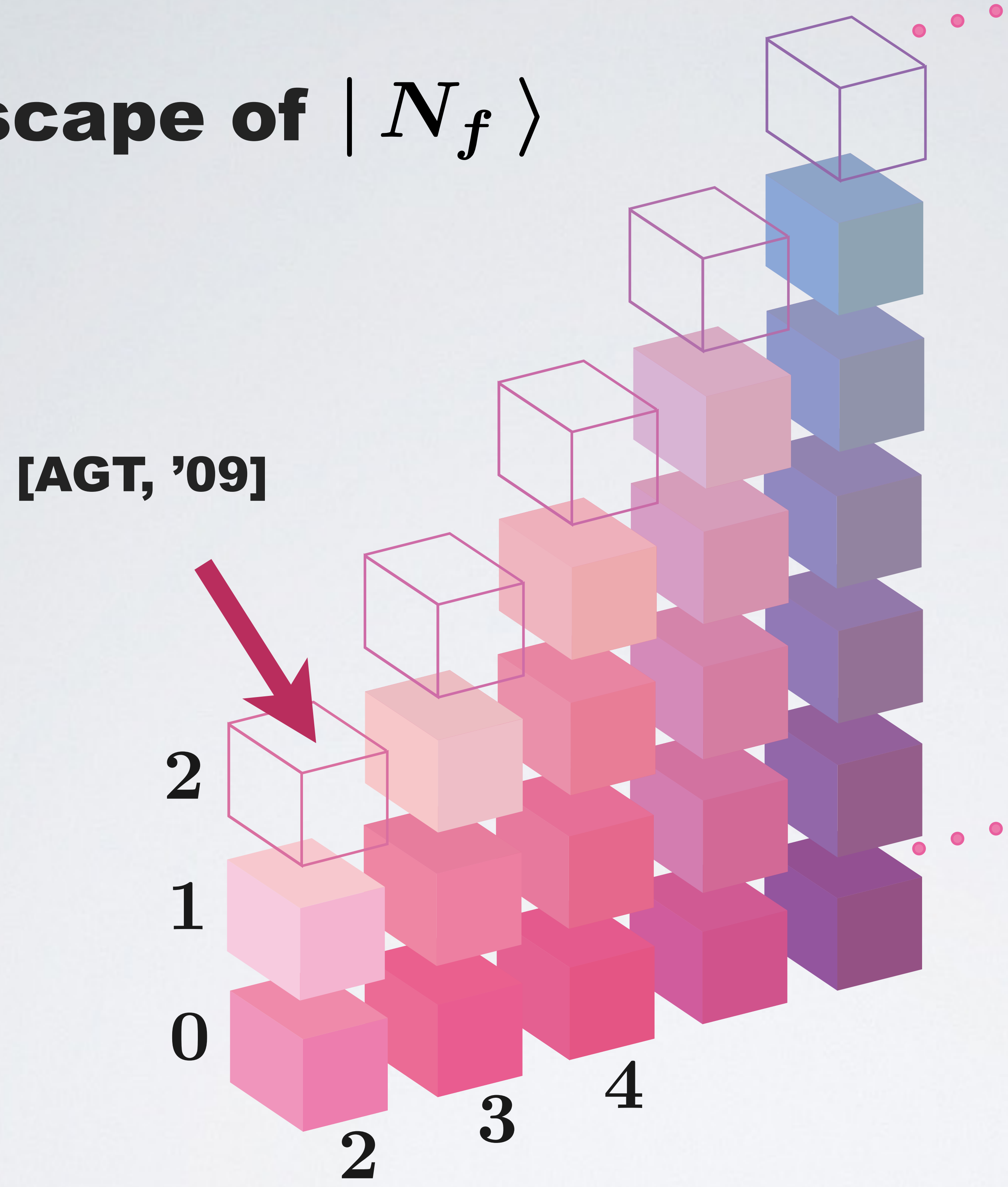

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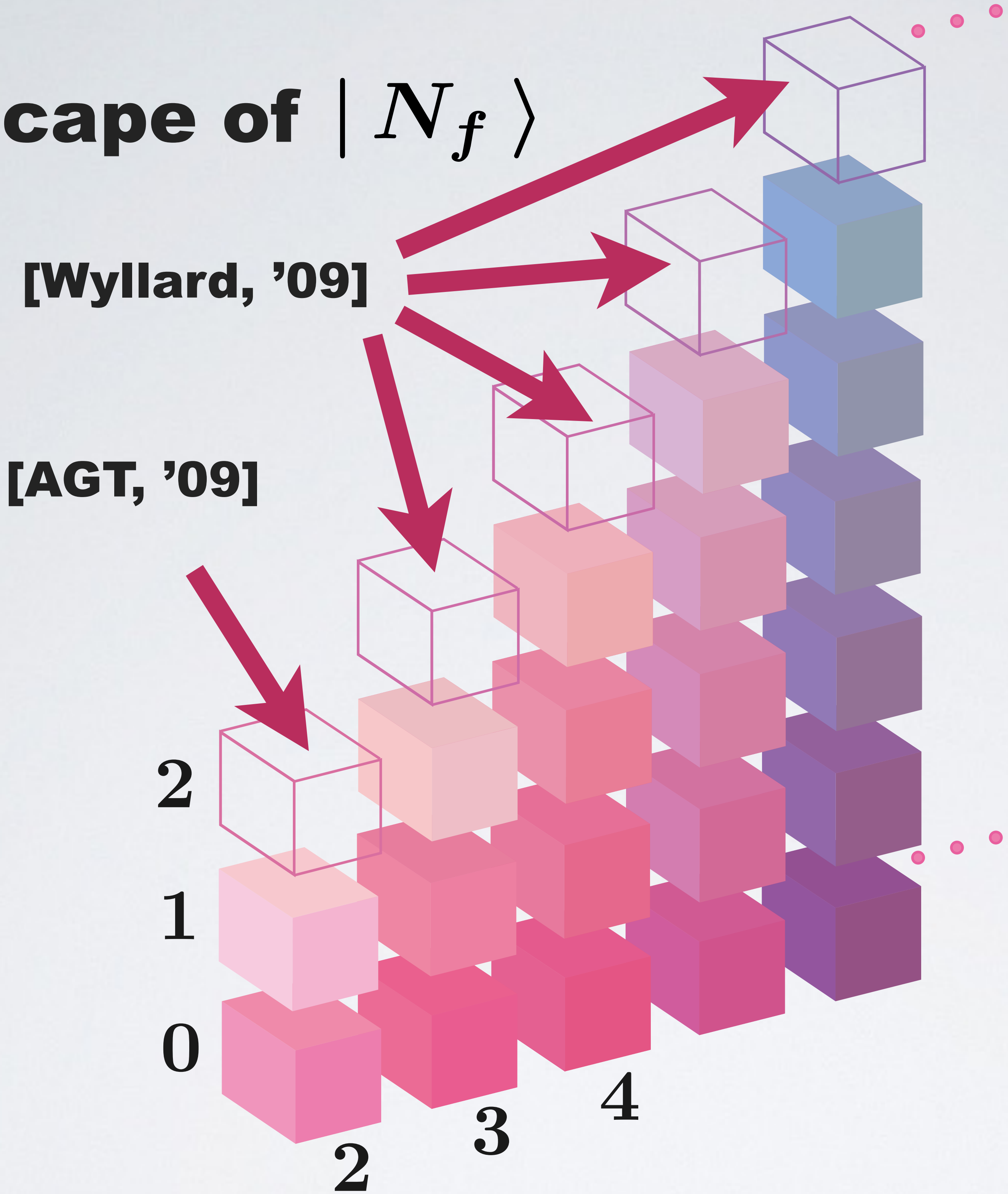

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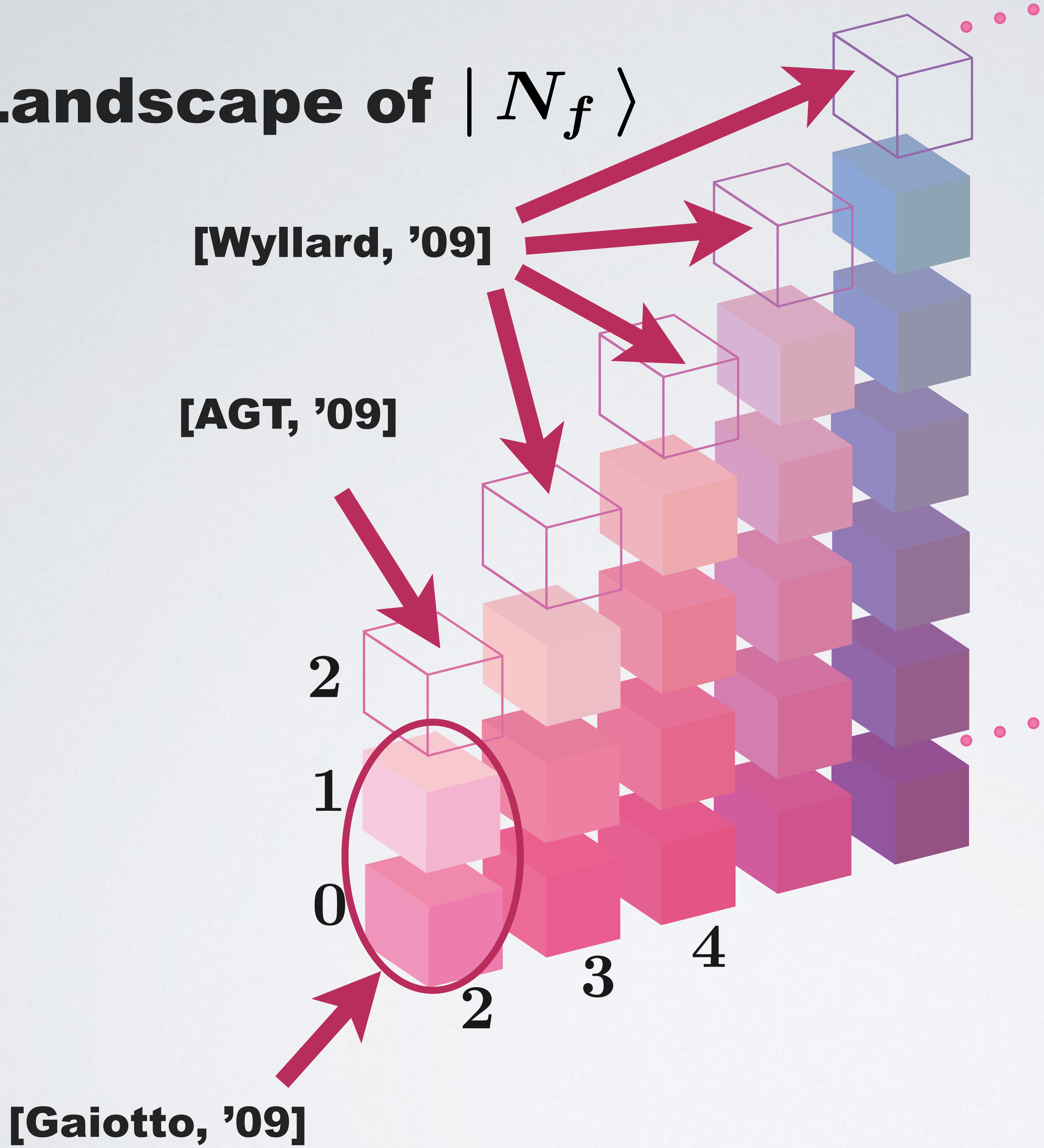

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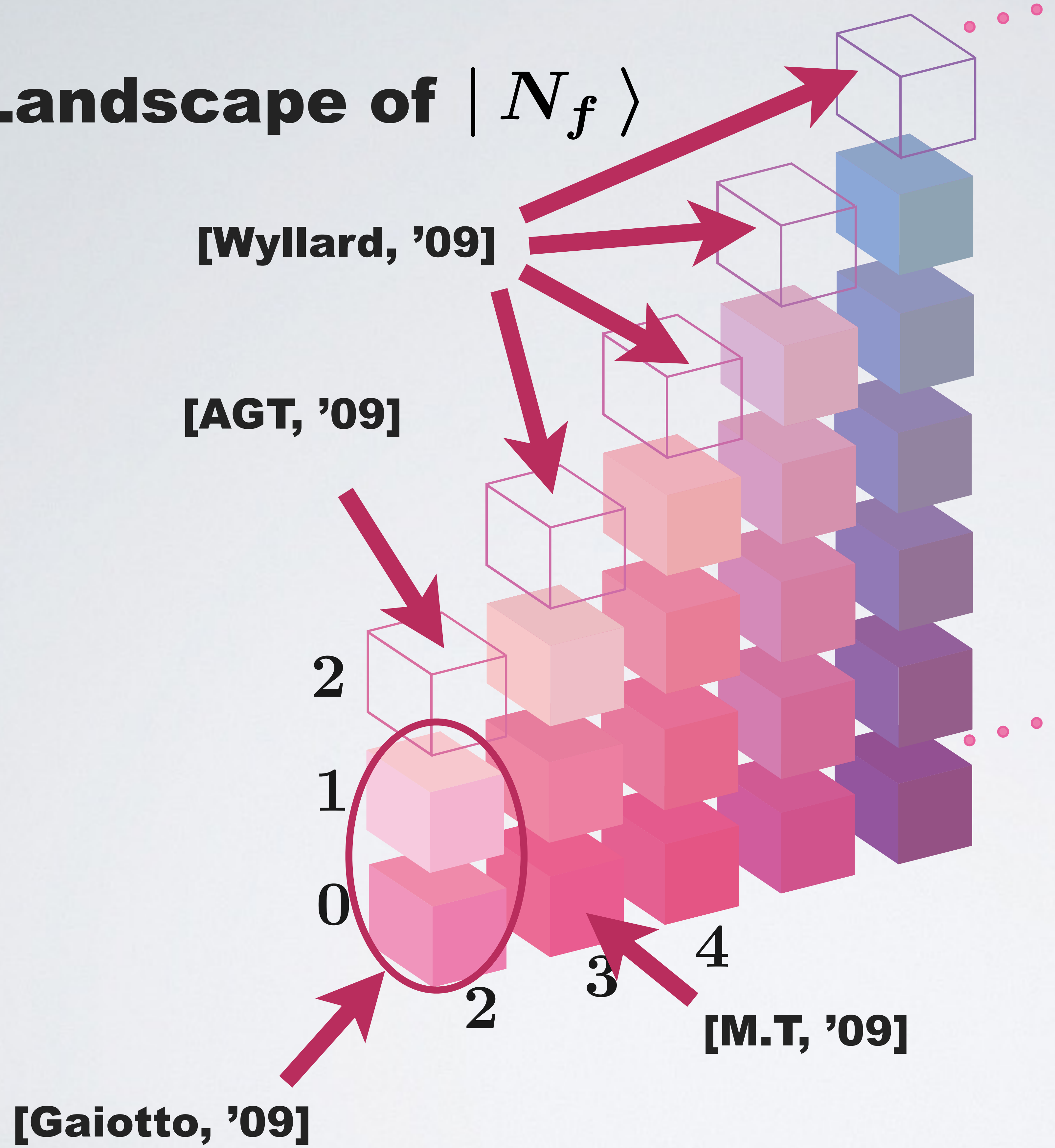
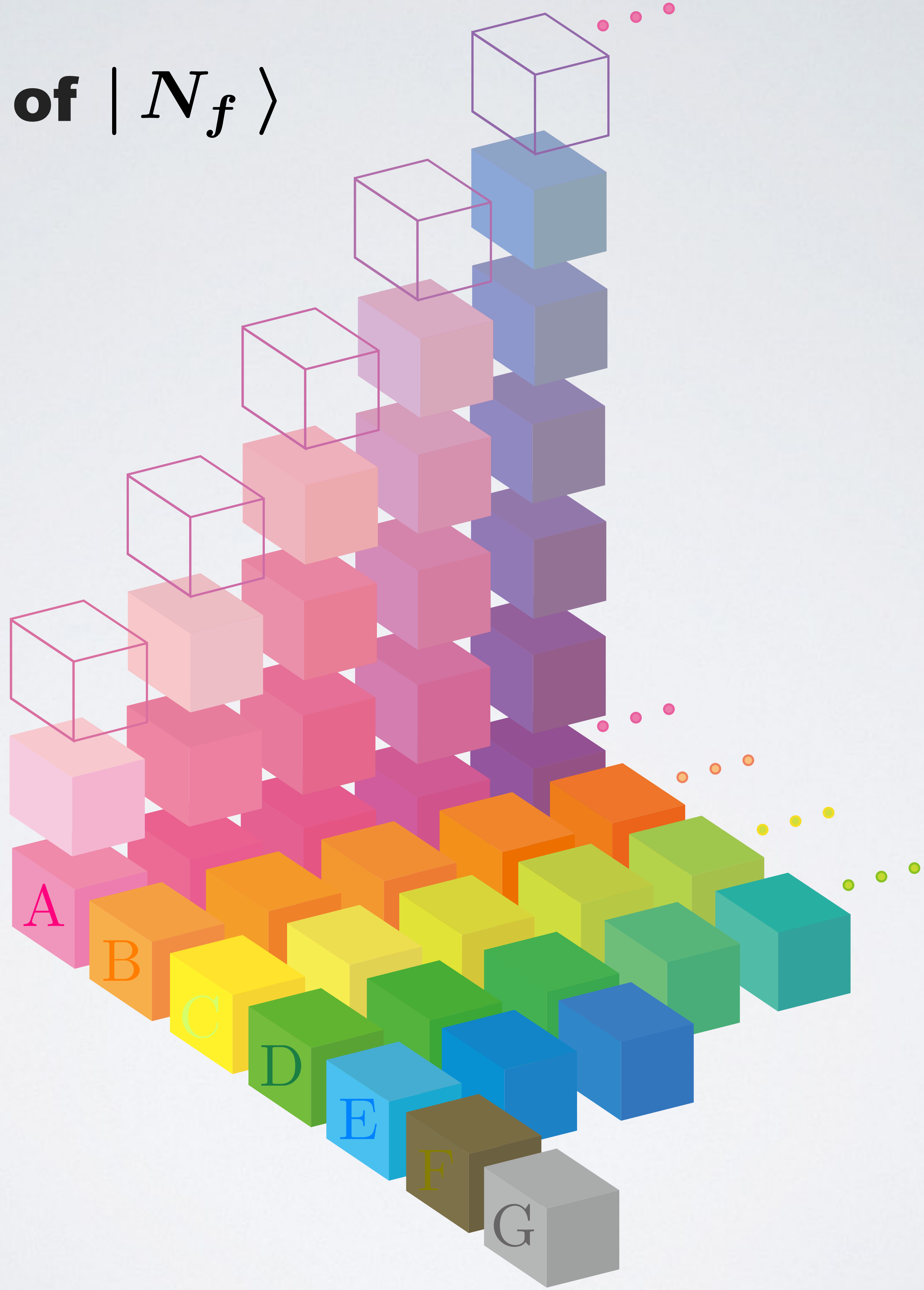
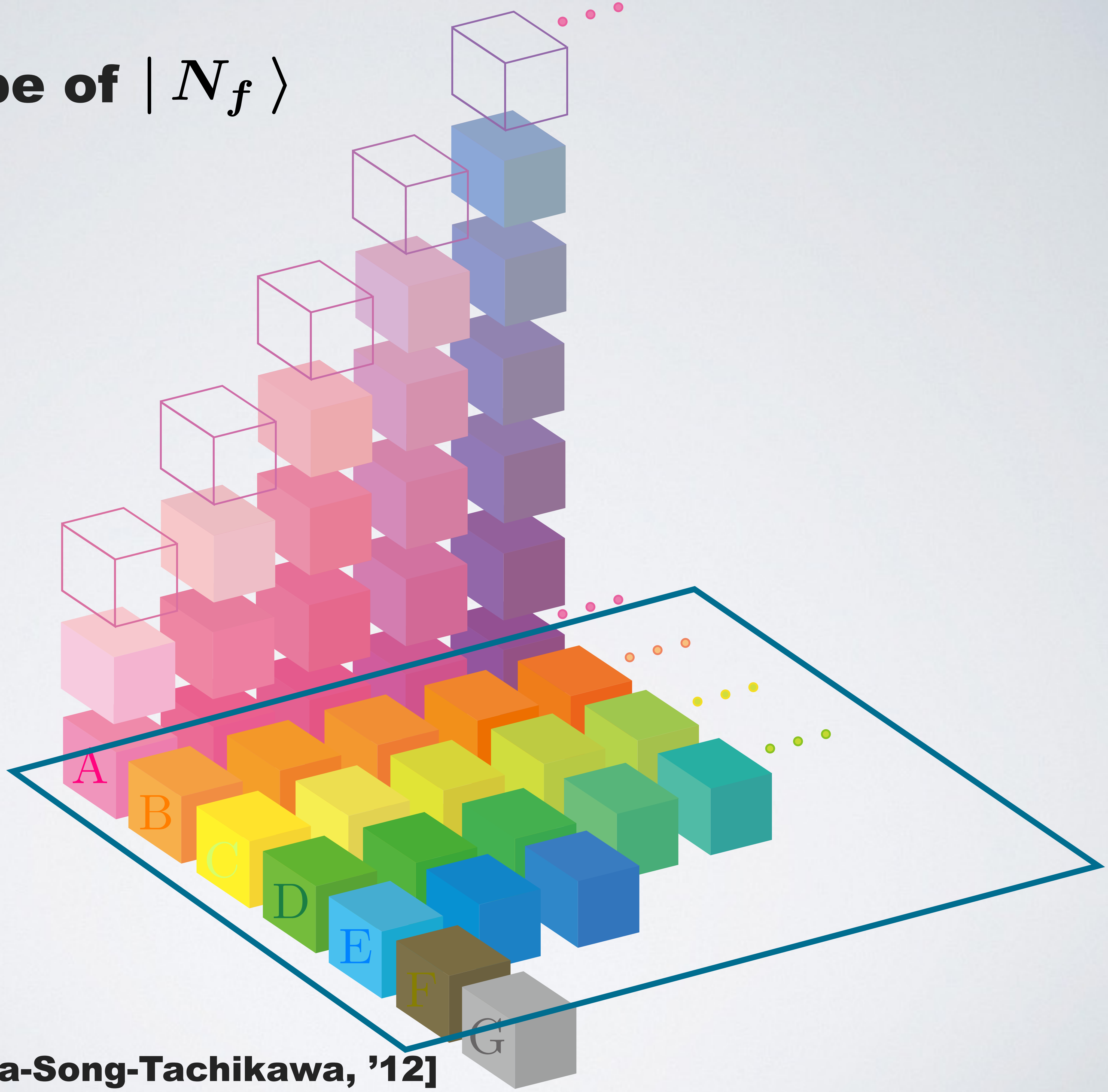


Landscape of $|\,N_f\,\rangle$



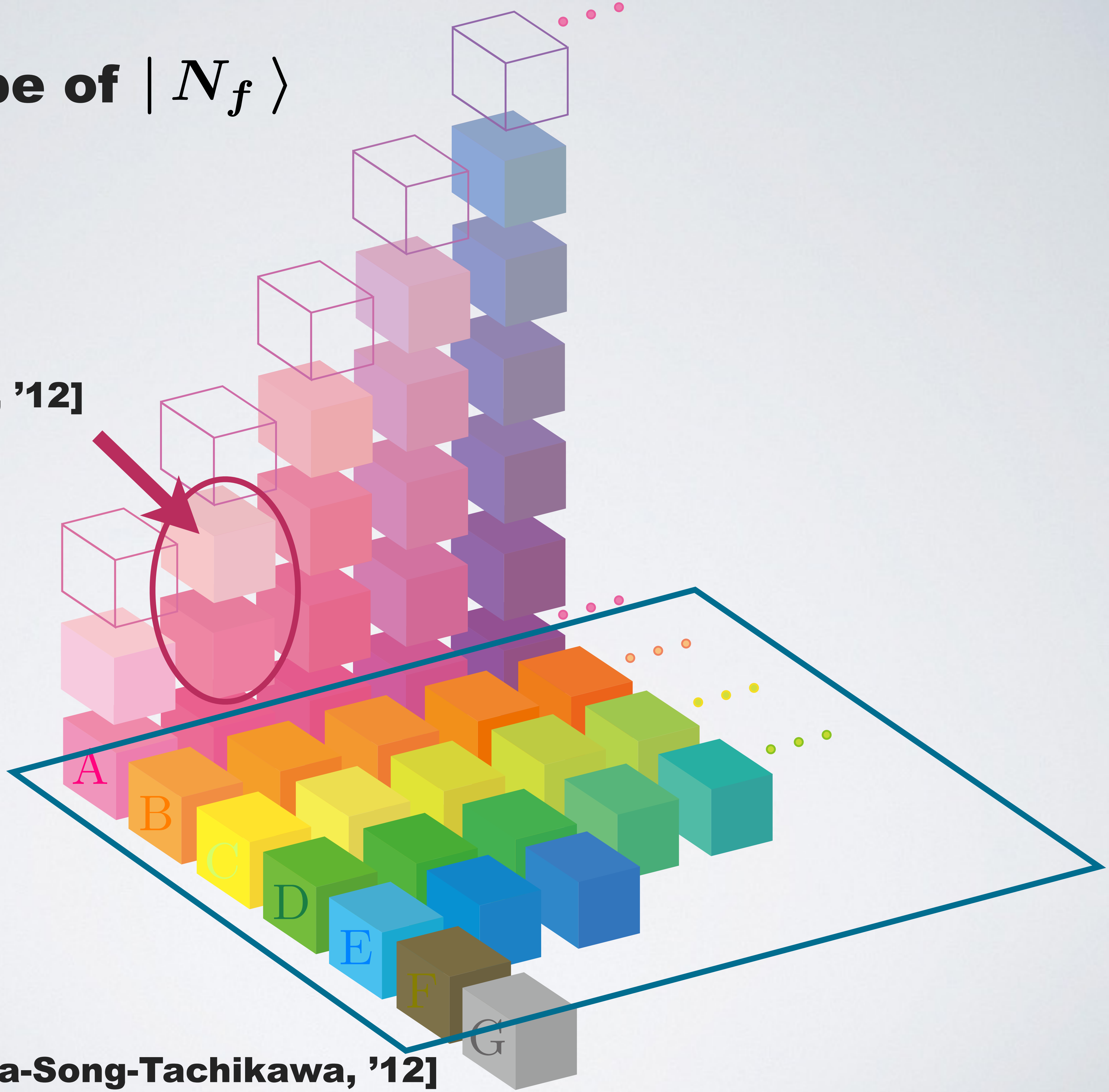
Landscape of $|\, N_f \,\rangle$



[Keller-Mekareeya-Song-Tachikawa, ’12]

Landscape of $|\, N_f \,\rangle$

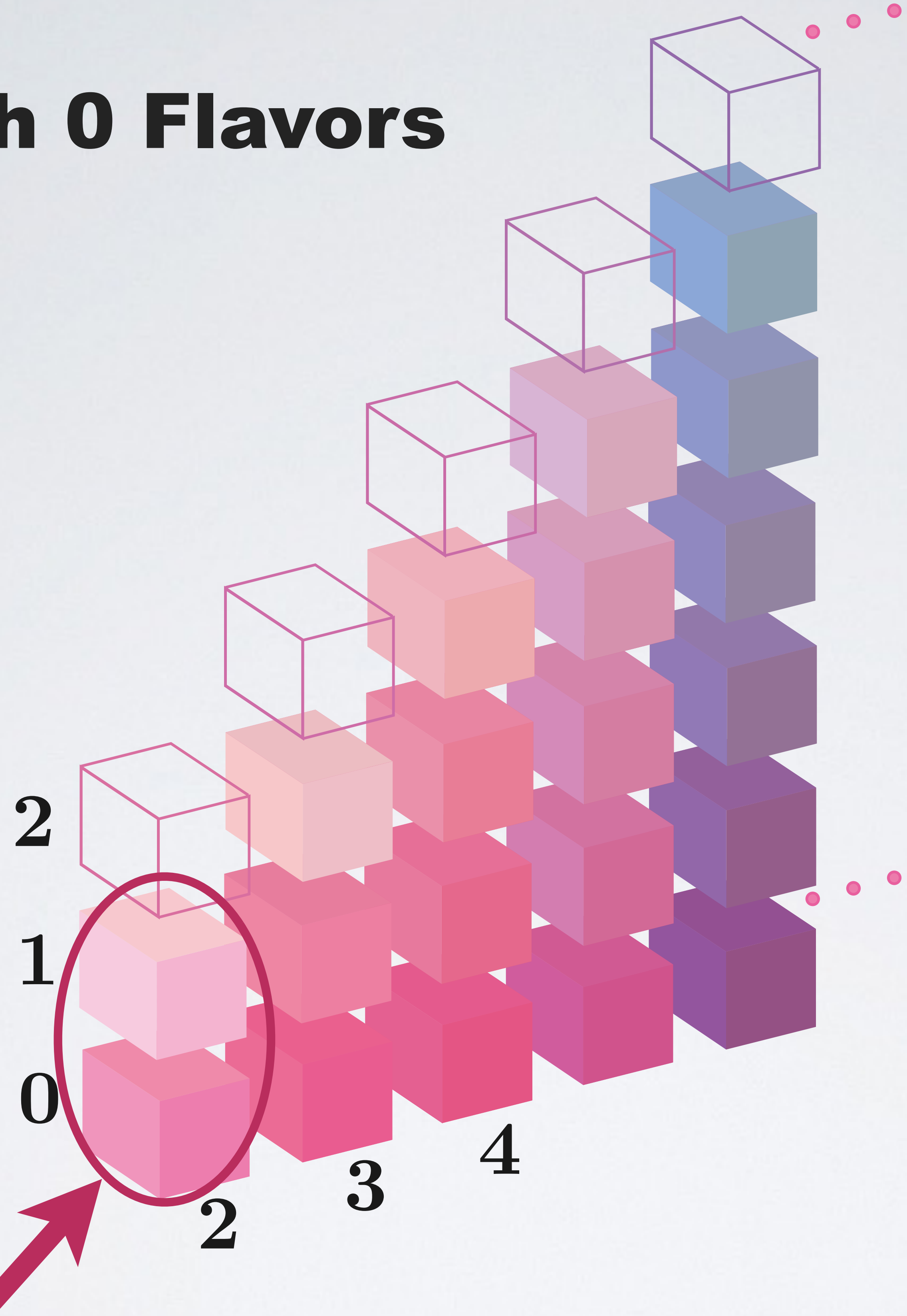
[Kanno-M.T, '12]



[Keller-Mekareeya-Song-Tachikawa, '12]

SU(2)

SU(2) with 0 Flavors



[Gaiotto, ’09]

SU(2) with 0 Flavors

$$N_f = 0$$

$$L_1 |\, 0 \,\rangle = |\, 0 \,\rangle \qquad L_2 |\, 0 \,\rangle = 0$$

SU(2) with 0 Flavors

$$N_f = 0$$

$$L_1 |\, 0 \,\rangle = |\, 0 \,\rangle \qquad\qquad L_2 |\, 0 \,\rangle = 0$$

* It means 0-flavor, **not** vacuum

SU(2) with 0 Flavors

$$N_f = 0$$

$$L_1 | 0 \rangle = | 0 \rangle \qquad L_2 | 0 \rangle = 0$$

$$Z_{SU(2)}^{N_f = 0} = \langle 0 | 0 \rangle$$

SU(2) with 1 Flavors

$$N_f = 1$$

$$L_1|1\rangle = 2m\Lambda|1\rangle \qquad L_2|1\rangle = \Lambda^2|1\rangle$$

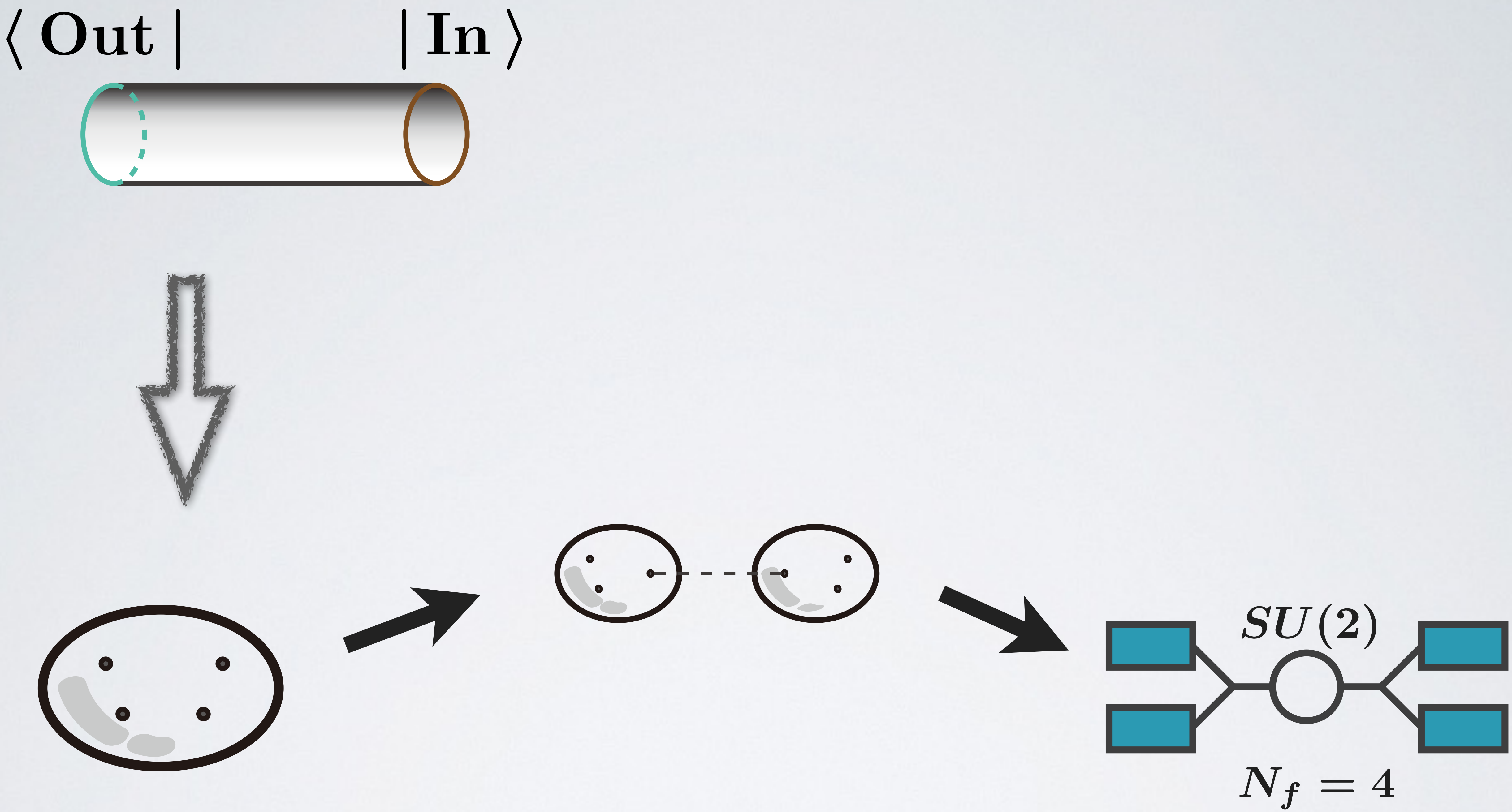
SU(2) with 1 Flavors

$$N_f = 1$$

$$L_1|1\rangle = 2m\Lambda|1\rangle \qquad L_2|1\rangle = \Lambda^2|1\rangle$$

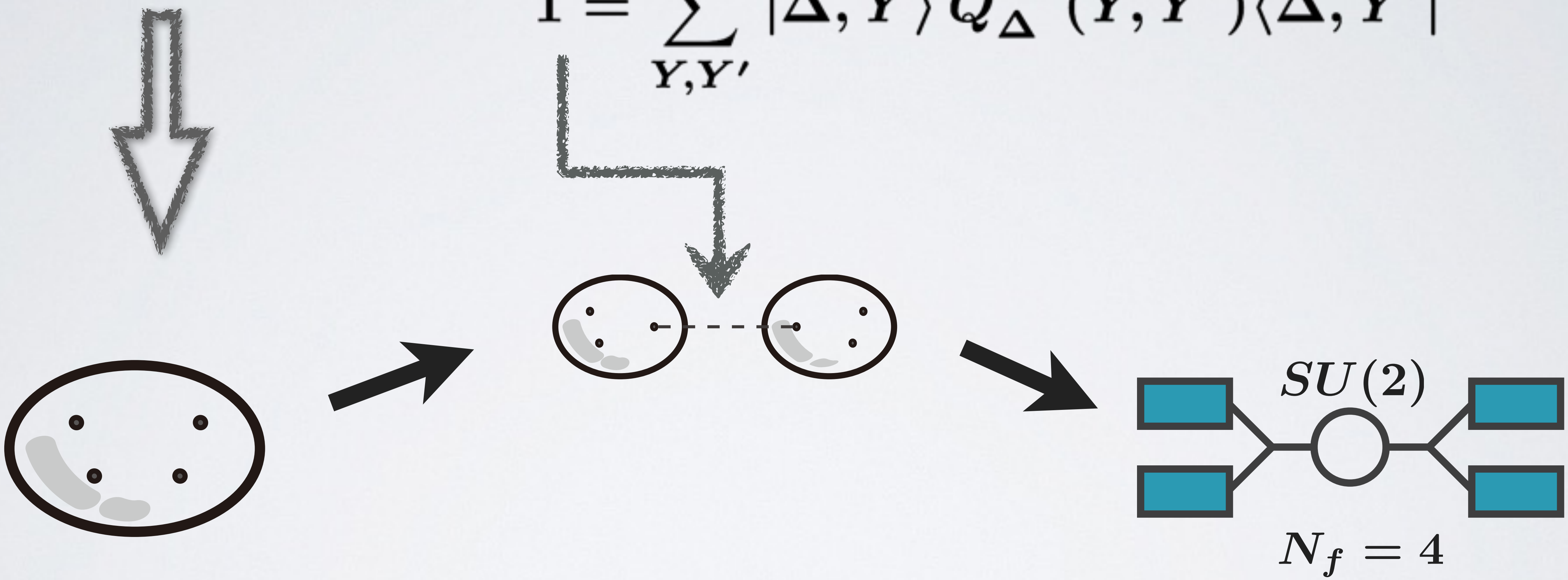
$$Z_{SU(2)}^{N_f=1} = \langle 0|1\rangle = \langle 1|0\rangle$$

4-flavors case is special (original AGT)


⟨ Out |
| In ⟩
$SU(2)$
$N_f = 4$

$\langle\, \mathbf{Out}\,|$ $\qquad\qquad$ $|\, \mathbf{In}\,\rangle$

$$\mathbf{1} = \sum_{Y,Y'} |\Delta, Y\rangle\, Q_{\Delta}^{-1}(Y, Y')\langle \Delta, Y'|$$



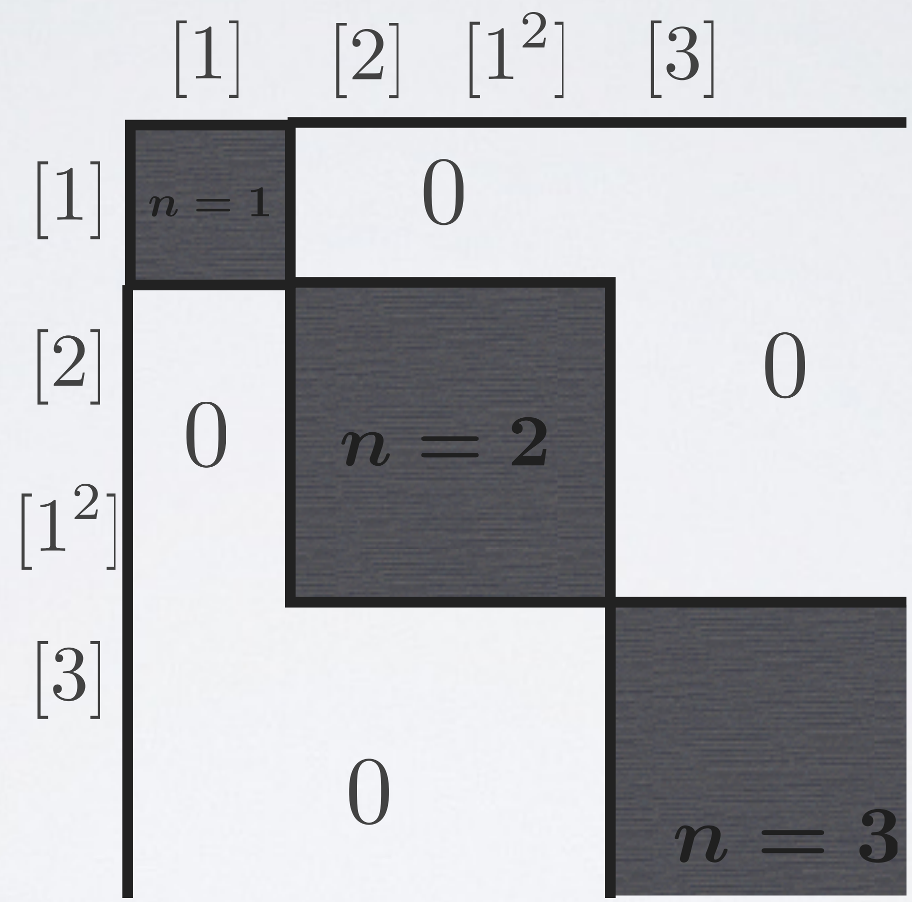
$$\langle \Delta_1 | V_2(1) V_3(q) | \Delta_4 \rangle = \langle \Delta_1 | V_2(1) \sum_{Y,Y'} |\Delta, Y\rangle Q_{\Delta}^{-1}(Y, Y')\langle \Delta, Y' | V_3(q) | \Delta_4 \rangle$$

the basis of the Verma module is not orthonormal

$$Q_{\Delta}(Y_1\,;\,Y_2) = \langle \Delta | L_{Y_2} L_{-Y_1} | \Delta \rangle$$

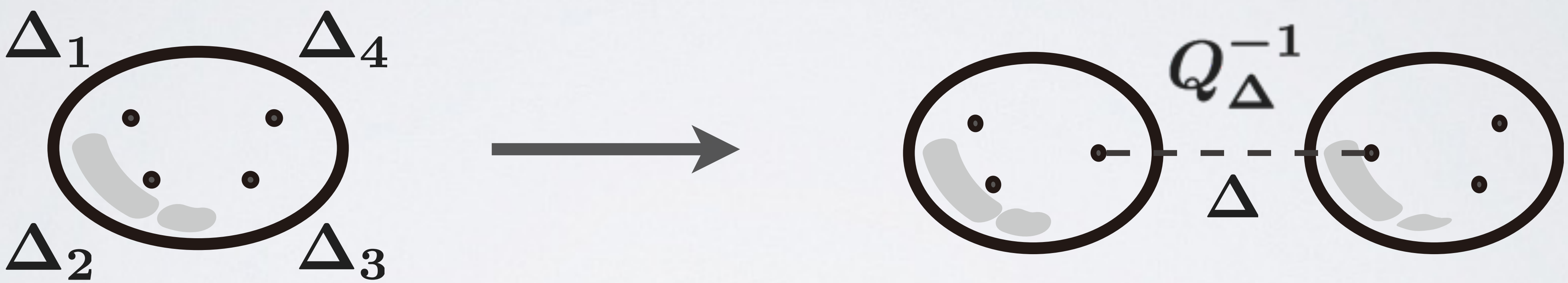
Shapovalov matrix

$$Q_{\Delta} =$$



$$\langle \Delta_1 | V_2(1) V_3(q) | \Delta_4 \rangle = \langle \Delta_1 | V_2(1) \sum_{Y,Y'} |\Delta, Y\rangle Q_\Delta^{-1}(Y,Y') \langle \Delta, Y' | V_3(q) | \Delta_4\rangle$$

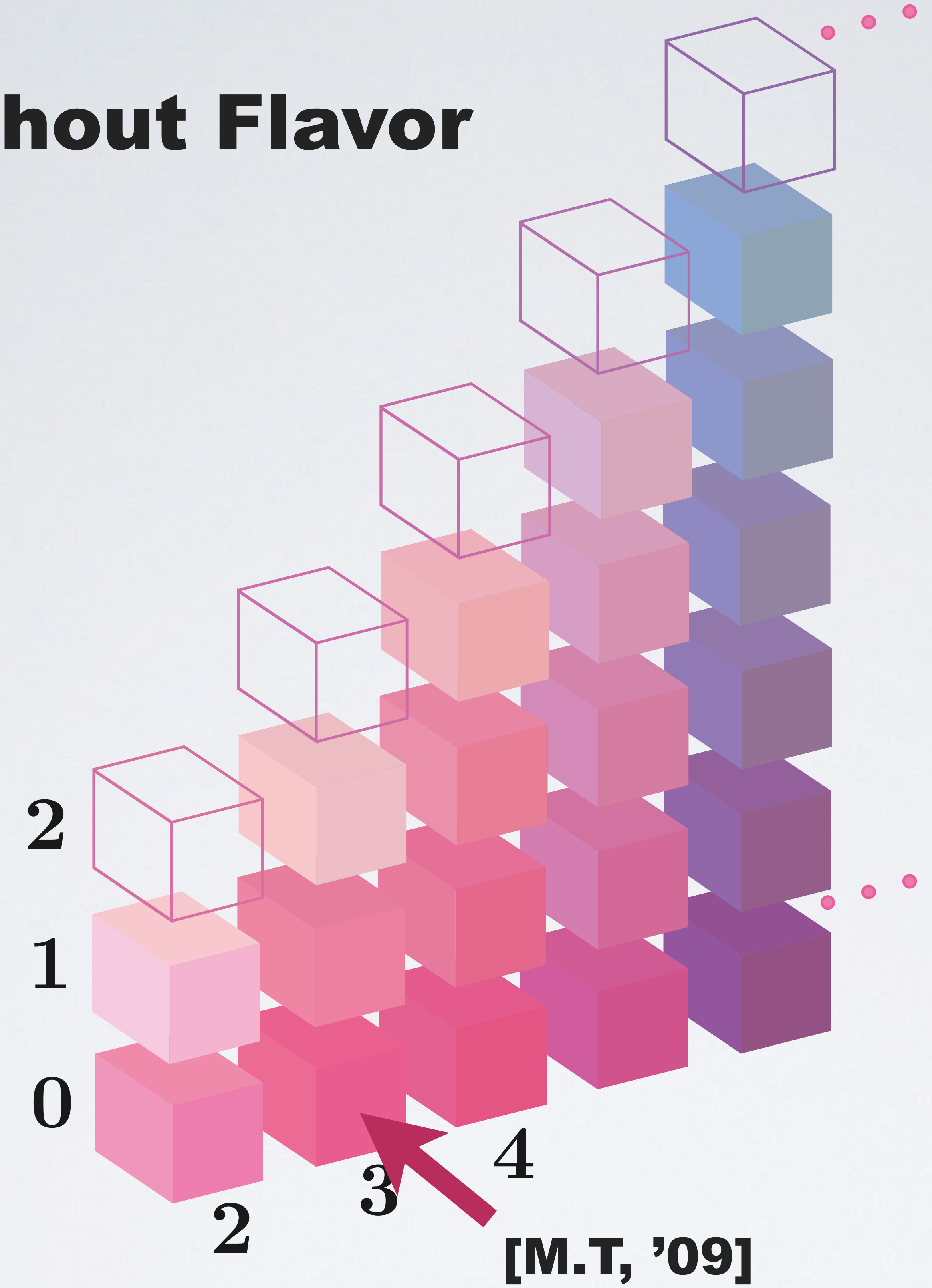
the basis of the Verma module is not orthonormal



$$= \sum_{\alpha} q^{\Delta(\alpha) - \Delta_1 - \Delta_2} \, b_{\Delta(\alpha)} \begin{bmatrix} \Delta_1 & \Delta_4 \\ \Delta_2 & \Delta_3 \end{bmatrix} \mathcal{B}_{\Delta(\alpha)} \begin{bmatrix} \Delta_1 & \Delta_4 \\ \Delta_2 & \Delta_3 \end{bmatrix} (q)$$

SU(3)

SU(3) without Flavor



SU(3) Whittaker state without Flavor

: theory with L_m and W_n

$$[L_m, W_n] = (2m - n)W_{n+m}$$

$$[W_m, W_n] =$$

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: theory with L_m and W_n

$$[L_m, W_n] = (2m - n) W_{n+m}$$

$$[W_m, W_n] =$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

$$[L_n, W_m] = (2n-m)W_{n+m}$$

$$[W_n, W_m] = \frac{9}{2}\Big[\frac{c}{3\cdot 5!}(n^2-1)(n^2-4)\delta_{n,-m} + \frac{16}{22+5c}(n-m)\Lambda_{n+m}$$

$$+(n-m)\left(\frac{(n+m+2)(n+m+3)}{15} - \frac{(n+2)(m+2)}{6}\right)L_{n+m}\Big]$$

$$\Lambda_n = \sum_{m\in\mathbb{Z}} : L_m L_{n-m} : + \frac{x_n}{5}L_n$$

non-linear algebra (~~Lie~~)

$$x_{2l} = (1-l)(1+l), \quad x_{2l+1} = (1-l)(12+l)$$

SU(3) Whittaker state without Flavor

$$N_f = 0$$

$$L_1 | 0 \rangle = 0 \qquad W_1 | 0 \rangle = | 0 \rangle$$

SU(3) Whittaker state without Flavor

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$$\boxed{Z_{SU(3)}^{N_f=0} = \langle 0 | 0 \rangle}$$

SU(3) Whittaker states with 0,1 Flavors

$$N_f = 0$$

$$L_1 | 0 \rangle = 0 \qquad W_1 | 0 \rangle = | 0 \rangle$$

$$N_f = 1$$

$$L_1 | 1 \rangle = | 1 \rangle \qquad W_1 | 1 \rangle = m | 1 \rangle$$

SU(3) Whittaker states with 0,1 Flavors

$$Z_{SU(3)}^{N_f=1} = \langle\, 0 \,|\, 1 \,\rangle = \langle\, 1 \,|\, 0 \,\rangle$$

$$Z_{SU(3)}^{N_f=2} = \langle\, 1 \,|\, 1 \,\rangle$$

SU(3) Whittaker states with 0,1 Flavors

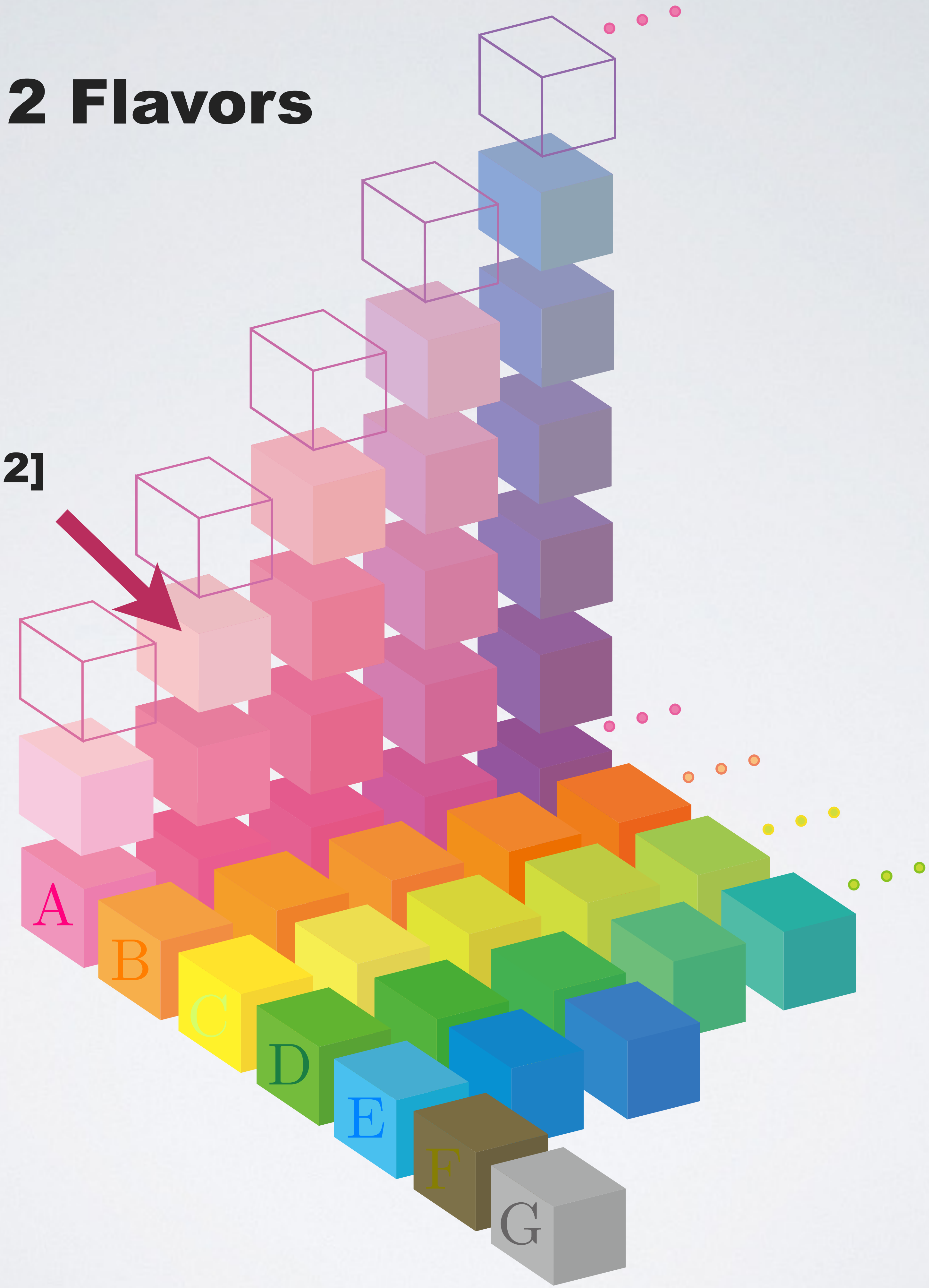
$$Z_{SU(3)}^{N_f=1} = \langle\, 0 \,|\, 1 \,\rangle = \langle\, 1 \,|\, 0 \,\rangle$$

$$Z_{SU(3)}^{N_f=2} = \langle\, 1 \,|\, 1 \,\rangle$$

$$Z_{SU(3)}^{N_f=2} = \langle\, 0 \,|\, 2 \,\rangle = \langle\, 2 \,|\, 0 \,\rangle \text{ ?}$$

SU(3) with 2 Flavors

[Kanno-M.T, ’12]



SU(3) with 2 Flavors → Trouble !?

SU(3) with 2 Flavors $\longrightarrow$ Trouble !?

Qestion.

$|\, 2\, \rangle$ **must be** L_1, L_2, W_2, W_3 **eigenstate.**

$$W_2 = [L_1, W_1]$$

But

$$3W_3 = [L_2, W_1]$$

SU(3) with 2 Flavors $\longrightarrow$ Trouble !?

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But

$$W_2 = [L_1, W_1]$$

$$3W_3 = [L_2, W_1]$$

Answer.

$$(W_1 + L_0)|\,2\,\rangle \propto |\,2\,\rangle$$

$$[L_n, L_0] = nL_n$$

generalized Whittaker states

$\{L_1, L_2\}$

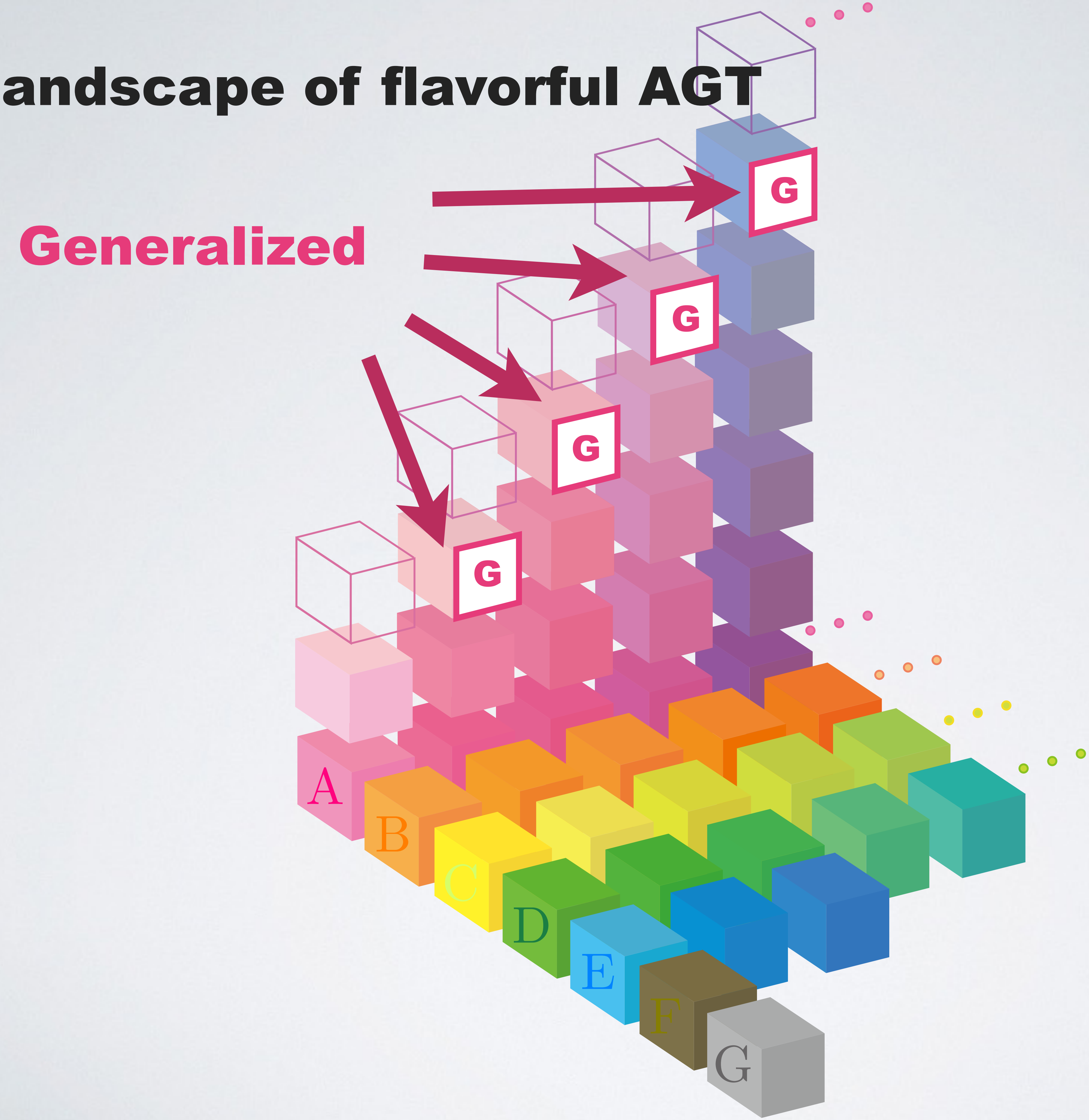
$\{L_0\}$: Cartan

$\}$ ← eigenstate of linear combi. of them

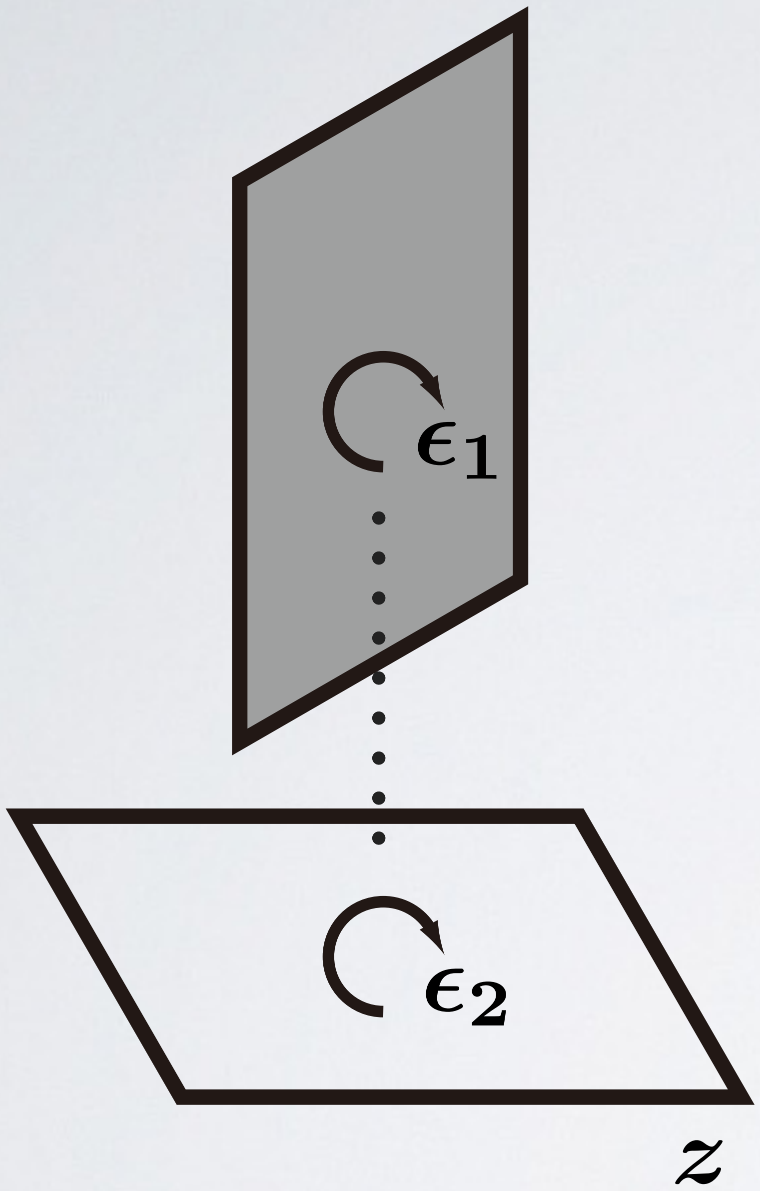
$\{L_{-1}, L_{-2}\}$

This is actually very ubiquitous B.C. for M5s !

Landscape of flavorful AGT



Surface defect



As a defect, a surface operator creates the singularity near the locus of it's insertion:

$$A \sim \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \frac{dz}{z}$$

$$\longrightarrow \begin{cases} m = \int_{\mathbb{C}_{z=0}} F \\ k = \int_{\mathbb{C}^2} F \wedge F \end{cases}$$

Instantons in the presence of such a are operator labelled by these two topological numbers.

This is a solution of the following modified SD eq.

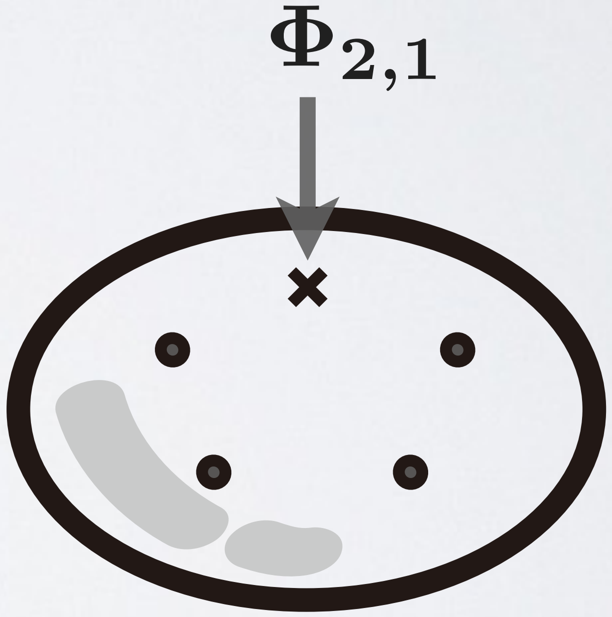
$$F + \star F = \omega (\delta_{\mathbb{C}})^{+}$$

Surface defect

surface operator ⟷ **insertion of degenerate field** $\Phi_{2,1}$

$$\Psi(z) = \exp\sqrt{\frac{\epsilon_1}{\epsilon_2}}\mathcal{G}(z) + \cdots$$

$$\langle\Delta_1|V_2(1)V_3(q)\Phi_{2,1}(z)|\Delta_4\rangle$$



$$(L_{-1}^2 - b^2 L_{-2})\Phi_{2,1}(z) = 0$$

Loop operators

$$W_R(L) = \mathrm{Tr}_R \, \mathrm{P} \, \exp g \oint_L iA$$

Wilson loop (electric)

Loop operators

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Wilson loop (electric)

$H_j(L) =$ **path integral with**

$$A \propto \begin{pmatrix} q/2 & 0 \\ 0 & -q/2 \end{pmatrix} d\Omega_2$$

t'Hoft loop (magnetic)

Loop operators

$$H_j(L) = \text{ path integral with}$$

$$A \propto \begin{pmatrix} q/2 & 0 \\ 0 & -q/2 \end{pmatrix} d\Omega_2$$

t'Hoft loop (magnetic)

Loop operators

monodromy

Wilson-'tHooft loop operator for the ele.-mag. charge (p,q)

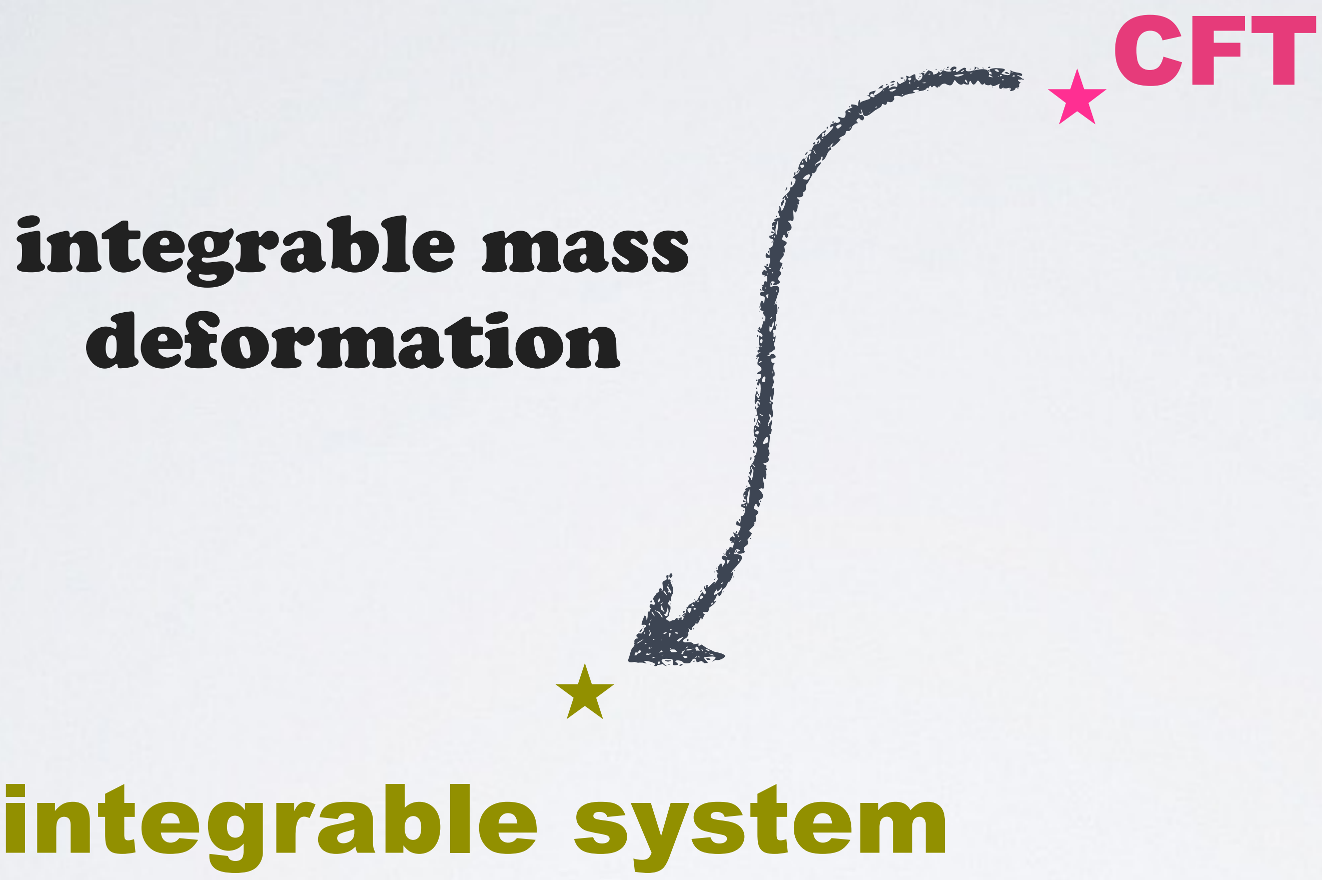
$$W_{\Box}(a) = \sum_{j=-1/2}^{1/2} e^{4\pi i j b(a-\epsilon/2)}$$

$$W_{\Box\Box}(a) = \sum_{j=-1}^{1} e^{4\pi i j b(a-\epsilon/2)}$$

[Alday-Gaoitto-Gukov-Tachikawa-Verlinde, '09]

[Drukker-Gomiz-Okuda-Techner, '09]

5d and 6d gauge theories



5d and 6d gauge theories

★ **CFT**

integrable mass deformation

Virasoro (W) algebras

★

integrable system

deformed Virasoro (W) algebras

q-deformed, elliptic, ...

proof of AGT

$$\mathcal{H}_{inst} = H^*(\mathcal{M})$$

geometric representation theory

→ **mathematically, AGT is proved (for special cases)**

5. Conclusion

Conclusion

AGT has a stringy origin: 6d -> 4d vs 2d

Many variations

(gauge group, matters, operators, dimensions ..)

Math proof, but no intuitive understandings