Recent developments of Functional Renormalization Group and its applications to ultracold fermions

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Today's contents

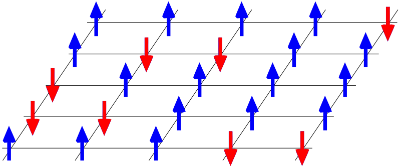
1. Introduction to renormalization group
2. Functional renormalization group
3. Applications

Introduction to renormalization group (RG)

Ising spin model

Hamiltonian for a spin system
on the lattice $a\mathbb{Z}^d$:

$$H = -\sum_{|i-j|=a} J_{ij} S_i S_j .$$



i, j: labels for lattice points
$S_i = \pm 1$: spin variable .

Let us consider the ferromagnetic case: $J_{ij} \geq 0$.

- Spins are aligned parallel $\Rightarrow$ Energy H takes lower values.

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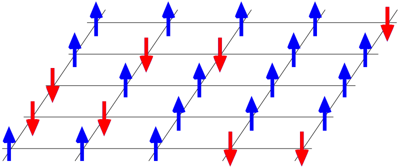

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- Spins are aligned parallel $\Rightarrow$ Energy H takes lower values.

What is properties of the following Gibbsian measure?

$$\mu(\{S_i\}) = \exp(-H/T + h\sum_i S_i)/Z.$$

($Z = \mathrm{Tr}\exp(-H/T + h\sum_i S_i)$: partition function)

Phase structure of the Ising model

Free energy of the system: $F(T,h) = -T \ln Z = -T \ln \mathrm{Tr} \exp(-H/T + h\sum_i S_i)$.

Rough estimate on $F(T,0)$:

$$F(T,0) \sim E - T \ln W.$$

(W: Number of spin alignments with $H(\{S_i\}) = E (:= \langle H \rangle)$)

- $T \to \infty$: The second term becomes dominant, and spins are randomized.
- $T \to 0$: The first term becomes dominant, and spins like to be aligned.

Phase structure of the Ising model

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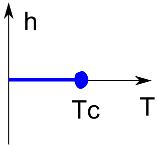
Phase structure of Ising spins:



For $T<T_c$, there exists discontinuities in $\partial F/\partial h$ when crossing the blue line (1st order PT line).

Magnetization

$$M(T,h)=\frac{\partial F(T,h)}{\partial h}=\langle S_i\rangle.$$

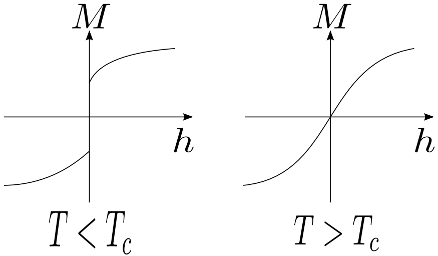
At $T<T_c$, M jumps as crossing $h=0$ (1st order phase transition.)

At $h=0$, naive Gibbsian measure μ is not well defined

$\Rightarrow$ The system must be specified with the boundary condition at infinities:

$$\mu(T,h=0,a)=a\mu_+(T)+(1-a)\mu_-(T)\qquad (0<a<1).$$

with $\mu_\pm(T)=\mu(T,h\to\pm 0)$



2nd order phase transition

At $T = T_c$, there exists no discontinuities on first derivatives of $F(T, h)$.

What about second derivatives? $\Rightarrow$ Magnetic susceptibility:

$$\chi = \frac{\partial^2 F}{\partial h^2} = \sum_i \langle S_0 S_i \rangle.$$

As $T \to T_c + 0$, the susceptibility diverges,

$$\chi \sim |(T - T_c)/T_c|^{-\gamma} \to \infty.$$

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Other scaling properties:

$$\begin{aligned} C &:= -(T - T_c)\frac{\partial^2 F}{\partial T^2} \sim |(T - T_c)/T_c|^{-\alpha} \to \infty, \\ M &\sim |(T - T_c)/T_c|^{\beta} \to 0. \end{aligned}$$

Scaling relation (Rushbrook identity):

$$\alpha + 2\beta + \gamma = 2.$$

(Only two of scaling exponents are independent!)

Scaling hypothesis

Assume that the free energy $G(T-T_c,h)=F(T,h)$ satisfies (Widom, 1965)

$$G(\lambda^s t,\lambda^r h)=G(t,h)/\lambda.$$

Taking derivatives of the both sides,

$$\begin{aligned} M(t,h) &= \lambda^{r+1}M(\lambda^s t,\lambda^r h),\\ \chi(t,h) &= \lambda^{2r+1}\chi(\lambda^s t,\lambda^r h),\\ C(t,h) &= \lambda^{2s+1}C(\lambda^s t,\lambda^r h). \end{aligned}$$

Putting $h=0$ and $\lambda=1/|t|^s$, we get

$$M(t,0)\sim|t|^{-(r+1)/s},\quad \chi(t,0)\sim|t|^{-(2r+1)/s},\quad C(t,0)\sim|t|^{-(2s+1)/s}.$$

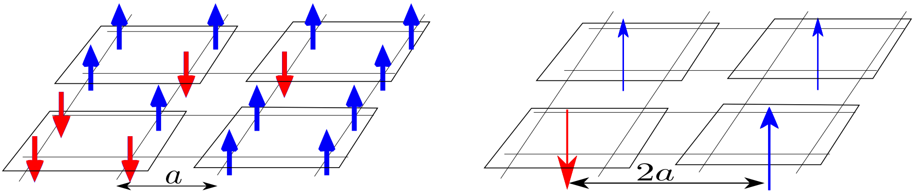
Then, $\alpha+2\beta+\gamma=2$ because

$$\alpha=\frac{2s+1}{s},\quad \beta=-\frac{r+1}{s},\quad \gamma=\frac{2r+1}{s}.$$

Block spin transformation (I)

What justifies scaling hypothesis? $\Rightarrow$ Renormalization Group

Block spin transformation (Kadanoff)



The number of total degrees of freedom: $N \mapsto N'$. This defines the scaling factor

$$\lambda = N'/N(< 1).$$

Idea: Long range feature is difficult to be computed from microscopic theory.

Important!

Let's perform coarse graining, and compute correlations between averaged spins!

Block spin transformation (II)

We can generally denote the BST as

$$\mu_R(\{S_i^{(R)}\}) = \sum_{\{S_i\}} T(\{S_i^{(R)}\}, \{S_i\})\mu(\{S_i\}).$$

T is a transition probability, which satisfies

$$\sum_{\{S_i^{(R)}\}} T(\{S_i^{(R)}\}, \{S_i\}) = 1.$$

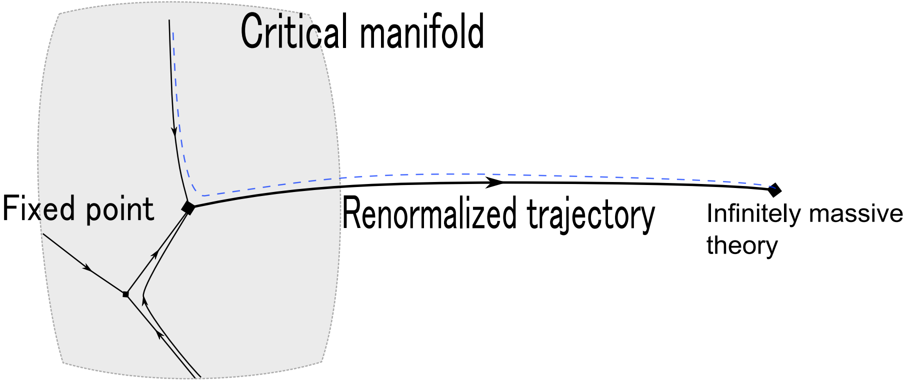
Assumption: μ_R is also Gibbsian with respect to block spins $S_i^{(R)}$: $\mu_R \propto \exp -H_R$.

$$H_R(S^{(R)}) = \mathcal{R}H(S) := -\ln \mu_R.$$

Free energy per unit cell: $F(H_R) = F(H)/\lambda$.

Renormalization group flow

Operation $\mathcal{R}$ can be performed repeatedly on effective Hamiltonians H.



Fixed points $H^* = \mathcal{R}H^* \Rightarrow$ The system shows universal/self-similar behaviors (Wilson).

Important!

RG provides a useful framework to extract and treat large-scale behaviors.

Functional renormalization group

General framework of FRG

Generating functional of connected Green functions:

$$\exp(W[J]) = \int \mathcal{D}\Phi \exp\left(-S[\Phi] + J \cdot \Phi\right).$$

infinite dimensional integration!

Possible remedy: Construct nonperturbative relations of Green functions!
($\Rightarrow$ Functional techniques)

- Dyson-Schwinger equations
- 2PI formalism
- Functional renormalization group (FRG)

Wilsonian renormalization group

Classical action/Hamiltonian: $S[\phi]=\int \mathrm{d}^d x\left[\frac{1}{2}(\nabla\phi)^2+\frac{m^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4\right]$.

"Block spins": $\phi_p=\int \mathrm{d}^d x e^{-\mathrm{i}px}\phi(x)$ for small momenta p.

Field theoretic formulation of RG (Wilson, Kogut 1974):

$$\exp -S_\Lambda[\phi] := \mathcal{N}_\Lambda \int \prod_{|p|\geq\Lambda} \mathrm{d}\phi_p \exp -S[\phi].$$

Important!

Correlation functions $\langle \phi_{p_1}\cdots\phi_{p_n}\rangle$ *with low momenta* $|p_i|\leq\Lambda$ *are calculable with* S_Λ *instead of the microscopic action* S.

Functional/Exact RG (I)

Schwinger functional W_k with an IR regulator R_k:

$$\exp(W_k[J]) = \int \mathcal{D}\phi \exp\left(-S[\phi] - \frac{1}{2}\phi \cdot R_k \cdot \phi + J \cdot \phi\right).$$

R_k: IR regulator, which controls low-energy excitations ($p^2 \leq k^2$).

k-derivatives of the both sides:

$$\begin{aligned} \partial_k \exp(W_k[J]) &= \int \mathcal{D}\phi - \frac{1}{2}\phi R_k \phi \exp\left(-S[\phi] - \frac{1}{2}\phi R_k \phi + J\phi\right) \\ &= -\frac{1}{2}\frac{\delta}{\delta J} R_k \frac{\delta}{\delta J} \exp W_k[J] \end{aligned}$$

Flow equation

$$\partial_k W_k = -\frac{1}{2}\frac{\delta W_k}{\delta J} R_k \frac{\delta W_k}{\delta J} - \frac{1}{2} R_k \frac{\delta^2 W_k}{\delta J \delta J}.$$

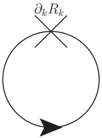
Functional/Exact RG (II)

The 1PI effective action Γ_k is introduced via the Legendre trans.:

$$\Gamma_k[\varphi] + \frac{1}{2}\varphi \cdot R_k \cdot \varphi = J[\varphi] \cdot \varphi - W_k[J[\varphi]],$$

which obeys the flow equation (Wetterich 1993, Ellwanger 1994, Morris 1994)

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \mathrm{STr} \frac{\partial_k R_k}{\delta^2 \Gamma_k[\Phi]/\delta\Phi\delta\Phi + R_k} =$$



Properties of Γ_k: $\Gamma_k \to S$ as $R_k \to \infty$, and $\Gamma_k \to \Gamma$ as $R_k \to 0$.

Important!

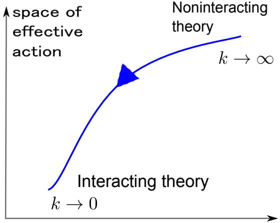
Functional implementation of "block spin transformations" keeps all the information of microscopic systems.

Generalized flow equation of FRG

$\delta S_k[\Phi]$: Some function of Φ with a parameter k. (IR regulator)

k-dependent Schwinger functional

$$\exp(W_k[J]) = \int \mathcal{D}\Phi \exp\left[-\left(S[\Phi] + \delta S_k[\Phi]\right) + J\cdot\Phi\right]$$



Flow equation

$$\begin{aligned} -\partial_k W_k[J] &= \langle \partial_k \delta S_k[\Phi] \rangle_J \\ &= \exp\left(-W_k[J]\right) \partial_k(\delta S_k) \left[\delta/\delta J\right] \exp(W_k[J]) \end{aligned}$$

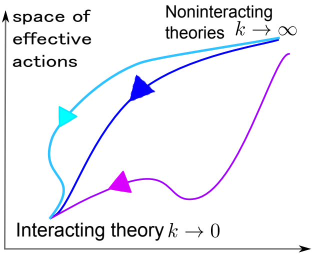
Consequence

We get a (functional) differential equation instead of a (functional) integration!

Optimization

Choice of IR regulators δS_k is arbitrary.

Optimization:
Choose the "best" IR regulator, which validates systematic truncation of an approximation scheme.



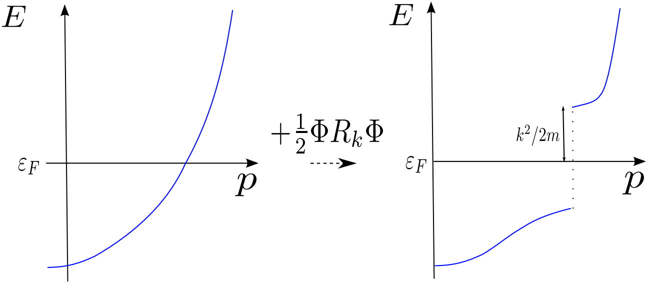
Optimization criterion (Litim 2000, Pawlowski 2007):

- IR regulators δS_k make the system gapped by a typical energy $k^2/2m$ of the parameter k.
- High-energy excitations $(\gtrsim k^2/2m)$ should decouple from the flow of FRG at the scale k.
- Choose δS_k stabilizing calculations and making it easier.

Conventional approach: Wetterich equation

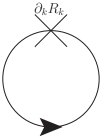
At high energies, perturbation theory often works well.
⇒ Original fields control physical degrees of freedom.

IR regulator for bare propagators (∼ mass term): $\delta S_k[\Phi] = \frac{1}{2}\Phi_\alpha R_k^{\alpha\beta}\Phi_\beta$.



Flow equation of 1PI effective action $\Gamma_k[\Phi]$ (Wetterich 1993)

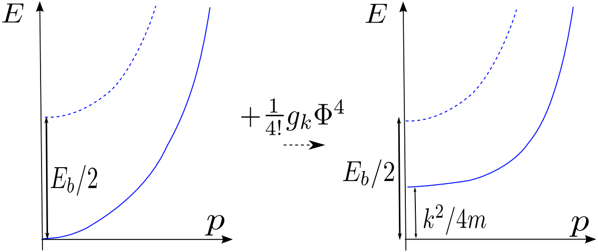
$$\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\frac{\partial_k R_k}{\delta^2\Gamma_k[\Phi]/\delta\Phi\delta\Phi + R_k} =$$



FRG beyond the naive one: vertex IR regulator

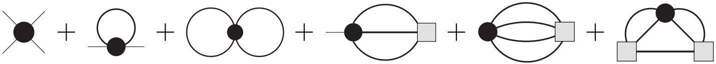
In the infrared region, collective bosonic excitations emerge quite in common.
(e.g.) Another low-energy excitation emerges in the $\Phi\Phi$ channel

Vertex IR regulator: $\delta S_k = \frac{1}{4!} g_k^{\alpha\beta\gamma\delta} \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta$.



Flow equation with the vertex IR regulator (YT, PTEP**2014**, 023A04)

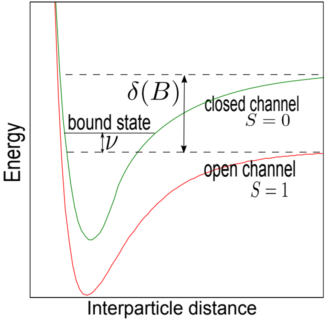
$\partial_k \Gamma_k[\Phi] =$

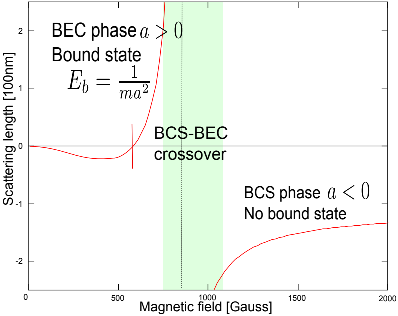


Application of fermionic FRG to the BCS-BEC crossover

Cold atomic physics

Ultracold fermions provides examples of strongly-correlated fermions.
High controllability can tune effective couplings with real experiments!



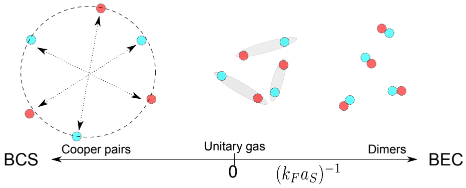


(Typically, $T\sim 100$nK, and $n\sim 10^{11-14}$ cm^{-3})

BCS-BEC crossover

EFT: Two-component fermions with an attractive contact interaction.

$$S = \int \mathrm{d}^4 x \left[\overline{\psi}(x)\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi(x) + g\overline{\psi}_1(x)\overline{\psi}_2(x)\psi_2(x)\psi_1(x)\right]$$



(Eagles 1969, Legget 1980,
Nozieres & Schmitt-Rink 1985)

Question

Is it possible to treat EFT systematically to describe the BCS-BEC crossover?

General strategy

We will calculate T_c/ε_F and μ/ε_F.
$\Rightarrow$ Critical temperature and the number density must be calculated.

We expand the 1PI effective action in the symmetric phase:

$$\Gamma_k[\overline{\psi},\psi] = \beta F_k(\beta,\mu) + \int_p \overline{\psi}_p[G^{-1}(p) - \Sigma_k(p)]\psi_p + \int_{p,q,q'} \Gamma_k^{(4)}(p)\overline{\psi}_{\uparrow,\frac{p}{2}+q}\overline{\psi}_{\downarrow,\frac{p}{2}-q}\psi_{\downarrow,\frac{p}{2}-q'}\psi_{\uparrow,\frac{p}{2}+q'}.$$

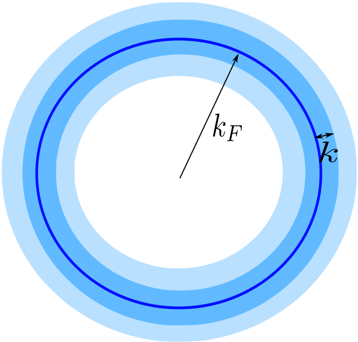
Critical temperature and the number density are determined by

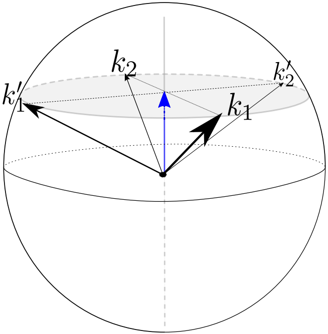
$$\frac{1}{\Gamma_0^{(4)}(p=0)} = 0, \qquad n = \int_p \frac{-2}{G^{-1}(p) - \Sigma_0(p)}.$$

BCS side

Case 1 Negative scattering length $(k_F a_s)^{-1} \ll -1$.
$\Rightarrow$ Fermi surface exists, and low-energy excitations are fermionic quasi-particles.

Shanker's RG for Fermi liquid (Shanker 1994)



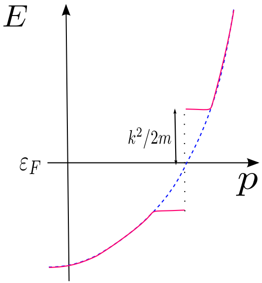


Functional implementation of Shanker's RG

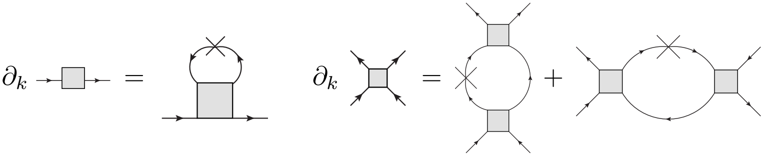
RG must keep low-energy fermionic excitations under control.

$\Rightarrow \delta S_k = \int_p \overline{\psi}_p R_k^{(f)}(\boldsymbol{p})\psi_p$ with

$$R_k^{(f)}(\boldsymbol{p}) = \mathrm{sgn}(\xi(\boldsymbol{p}))\left(\frac{k^2}{2m} - |\xi(\boldsymbol{p})|\right)\theta\left(\frac{k^2}{2m} - |\xi(\boldsymbol{p})|\right)$$

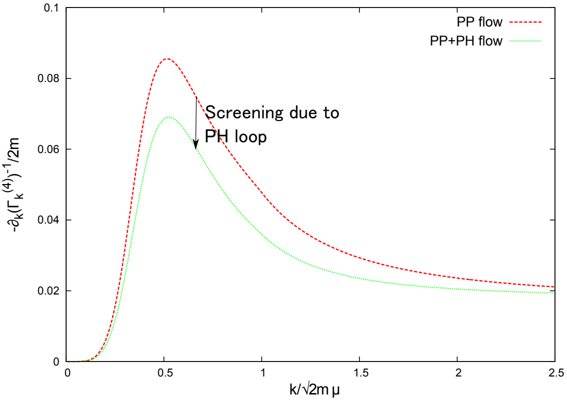


Flow equation of the self-energy Σ_k and the four-point 1PI vertex $\Gamma_k^{(4)}$:



Flow of fermionic FRG: effective four-fermion interaction

- Particle-particle loop ⇒ RPA & BCS theory
- Particle-hole loop gives screening of the effective coupling at $k \sim k_F$

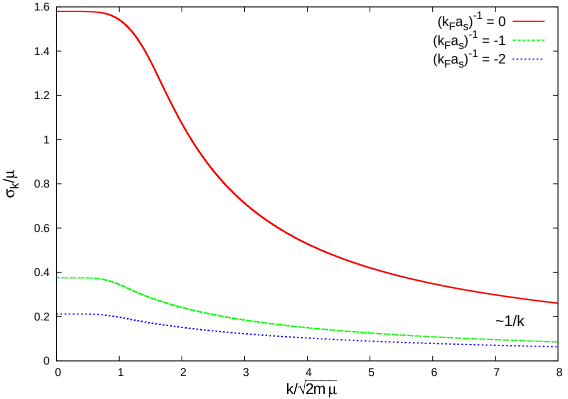


(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)

$T_c^{\rm BCS} = \varepsilon_F \frac{8e^{\gamma_E-2}}{\pi} e^{-\pi/2k_F|a_s|} \Rightarrow T_c^{\rm BCS}/2.2.$ (Gorkov, Melik-Barkhudarov, 1961)

Flow of fermionic FRG: self-energy

Local approximation on self-energy: $\Sigma_k(p) \simeq \sigma_k$.

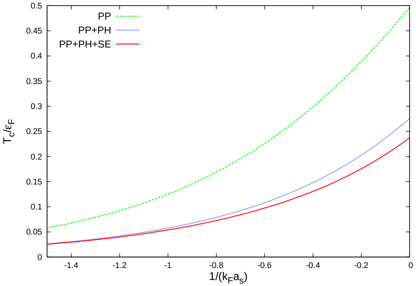


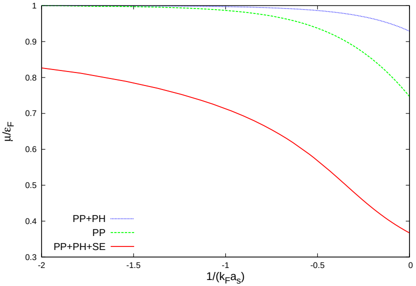
(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)

- High energy: $\sigma_k \simeq$ (effective coupling)$\times$(number density) $\sim 1/k$
- Low energy: $\partial_k \sigma_k \sim 0$ due to the particle-hole symmetry.

Transition temperature and chemical potential in the BCS side

(YT, G. Fejős, T. Hatsuda, arXiv:1310.5800)



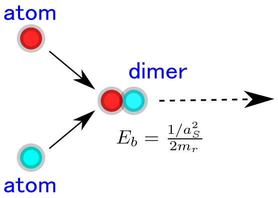


Consequence

- *Critical temperature T_c/ε_F is significantly reduced by a factor 2.2 in $(k_F a_s)^{-1} \lesssim -1$, and the self-energy effect on it is small in this region.*
- *$\mu(T_c)/\varepsilon_F$ is largely changed from 1 even when $(k_F a_s)^{-1} \lesssim -1$.*

BEC side

Case 2 Positive scattering length : $(k_F a_s)^{-1} \gg 1$
$\Rightarrow$ Low-energy excitations are one-particle excitations of composite dimers.



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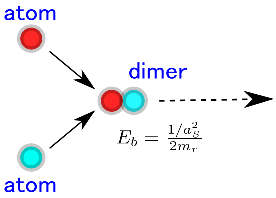


Several approaches for describing BEC of composite bosons. (Pros/Cons)

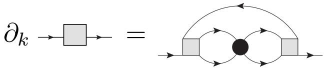
- Auxiliary field method
  (Easy treatment within MFA/ Fierz ambiguity in their introduction)
- Fermionic FRG ($\Leftarrow$ We develop this method!)
  (Unbiased and unambiguous/ Nonperturbative treatment is necessary)

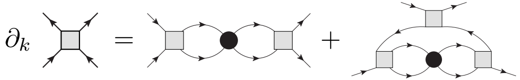
Vertex IR regulator & Flow equation

Optimization can be satisfied with the vertex IR regulator:

$$\delta S_k = \int_p \frac{g^2 R_k^{(b)}(\boldsymbol{p})}{1 - g R_k^{(b)}(\boldsymbol{p})} \int_{q,q'} \overline{\psi}_{\uparrow, \frac{p}{2}+q} \overline{\psi}_{\downarrow, \frac{p}{2}-q} \psi_{\downarrow, \frac{p}{2}-q'} \psi_{\uparrow, \frac{p}{2}+q'}$$

Flow equation up to fourth order (YT, PTEP**2014** 023A04, YT, arXiv:1402.0283):





Effective boson propagator in the four-point function:

$$\frac{1}{\Gamma_k^{(4)}(p)} = -\frac{m^2 a_s}{8\pi} \left(i p^0 + \frac{\boldsymbol{p}^2}{4m} \right) - R_k^{(b)}(\boldsymbol{p})$$

Flow of fermionic FRG: self-energy

Flow equation of the self-energy:

$$\partial_k \Sigma_k(p) = \int_l \frac{\partial_k \Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2 - \Sigma_k(l)}.$$

If $|\Sigma_k(p)| \ll 1/2ma_s^2$,

$$\Sigma_k(p) \simeq \int_l \frac{\Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2}$$

$$\simeq \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{(8\pi/m^2a_s) n_B(\boldsymbol{q}^2/4m + \frac{m^2a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}))}{ip^0 + \frac{\boldsymbol{q}^2}{4m} + \frac{m^2a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}) - \frac{(\boldsymbol{q}+\boldsymbol{p})^2}{2m} - \frac{1}{2ma_s^2}}.$$

Estimate of $|\Sigma_k(p)|$:

$$|\Sigma_k(p)| \lesssim \frac{1}{2ma_s^2} \times (\sqrt{2mT}a_s)^3 \times n_B(k^2/4m).$$

$\Rightarrow$ Our approximation is valid up to $(k^2/2m)/T \sim (k_F a_s)^3 \ll 1$.

Critical temperature in the BEC side

Number density:

$$n = \int_p \frac{-2}{ip^0 + \boldsymbol{p}^2/2m + 1/2ma_s^2 - \Sigma_0(p)} \simeq \frac{(2mT_c)^{3/2}}{\pi^2}\sqrt{\frac{\pi}{2}}\zeta(3/2).$$

Critical temperature and chemical potential:

$$T_c/\varepsilon_F = 0.218, \qquad \mu/\varepsilon_F = -1/(k_F a_s)^2.$$

⇒ Transition temperature of BEC.

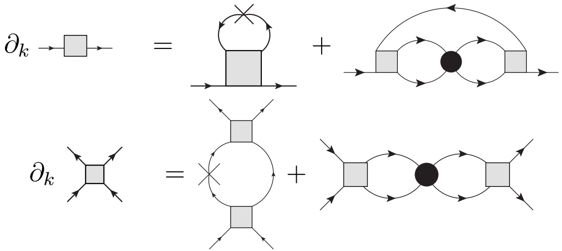
Consequence

FRG with vertex regulator provides a nonperturbative description of many-body composite particles.

fermionic FRG for the BCS-BEC crossover

We discuss the whole region of the BCS-BEC crossover with fermionic FRG.
$\Rightarrow$ Combine two different formalisms appropriate for BCS and BEC sides.

Minimal set of the flow equation for Σ_k and $\Gamma_k^{(4)}$:(YT, arXiv:1402.0283)



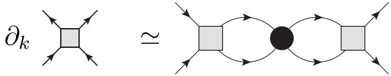
Flow of fermionic FRG with multiple regulators

Flow of four-point vertex:

Important property: fermions decouple from RG flow at the low energy region.

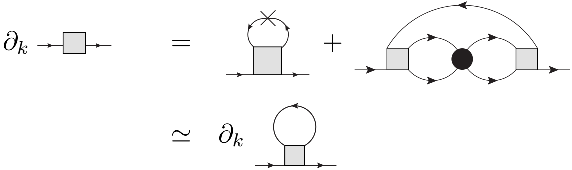
- In BCS side, fermions decouples due to Matsubara freq. ($k^2/2m \lesssim \pi T$).
- In BEC side, fermions decouples due to binding E. ($k^2/2m \lesssim 1/2ma_s^2$).

Approximation on the flow of the four-point vertex at low energy:



Flow of self-energy:

At a low-energy region, the above approx. gives



Qualitative behaviors of the BCS-BEC crossover from f-FRG

Approximations on the flow equation have physical interpretations.

Four-point vertex: Particle-particle RPA. The Thouless criterion $1/\Gamma^{(4)}(p=0)=0$ gives

$$\frac{1}{a_s} = -\frac{2}{\pi}\int_0^\infty \sqrt{2m\varepsilon}\mathrm{d}\varepsilon \left[\frac{\tanh\frac{\beta}{2}(\varepsilon-\mu)}{2(\varepsilon-\mu)} - \frac{1}{2\varepsilon}\right]$$

$\Rightarrow$ BCS gap equation at $T=T_c$.

Number density: $n=-2\int 1/(G^{-1}-\Sigma)$.

$$n=-2\int_p^{(T)} G(p) - \frac{\partial}{\partial\mu}\int_p^{(T)} \ln\left[1+\frac{4\pi a_s}{m}\left(\Pi(p)-\frac{m\Lambda}{2\pi^2}\right)\right].$$

$\Rightarrow$ Pairing fluctuations are taken into account. (Nozieres, Schmitt-Rink, 1985)

Consequence

We established the fermionic FRG which describes the BCS-BEC crossover.

Summary

Summary

- RG provides a useful framework to extract and treat large-scale behaviors.
- Functional implementation of coarse graining provides systematic treatment of field theories.
- Fermionic FRG is a promising formalism for interacting fermions.
  $\Rightarrow$ Separation of energy scales can be realized by optimization.
  $\Rightarrow$ Very flexible form for various approximation schemes.
- Fermionic FRG is applied to the BCS-BEC crossover.
  $\Rightarrow$ BCS side: GMB correction + the shift of Fermi energy from μ.
  $\Rightarrow$ BEC side: BEC without explicit bosonic fields.
  $\Rightarrow$ whole region: Crossover physics is successfully described at the quantitative level with a minimal setup on f-FRG.