Recent developments of Functional Renormalization Group

and

its applications to ultracold fermions

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Introduction to renormalization group (RG)
Introduction

Ising spin model

Hamiltonian for a spin system on the lattice $a\mathbb{Z}^d$:

$$H = - \sum_{|i-j|=a} J_{ij} S_i S_j.$$  

$i, j$: labels for lattice points  

$S_i = \pm 1$: spin variable.

Let us consider the ferromagnetic case: $J_{ij} \geq 0$.

- Spins are aligned parallel $\Rightarrow$ Energy $H$ takes lower values.

What is properties of the following Gibbsian measure?

$$\mu(\{S_i\}) = \frac{\exp(-H/T + h \sum_i S_i)}{Z}.$$  

$(Z = \text{Tr} \exp(-H/T + h \sum_i S_i)$: partition function)
Ising spin model

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Phase structure of the Ising model

Free energy of the system: \( F(T, h) = -T \ln Z = -T \ln \text{Tr} \exp\left(-H/T + h \sum_i S_i\right) \).

Rough estimate on \( F(T, 0) \):

\[
F(T, 0) \sim E - T \ln W.
\]

(\( W \): Number of spin alignments with \( H(\{S_i\}) = E := \langle H \rangle \))

- \( T \to \infty \): The second term becomes dominant, and spins are randomized.
- \( T \to 0 \): The first term becomes dominant, and spins like to be aligned.
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- \( T \to 0 \): The first term becomes dominant, and spins like to be aligned.

Phase structure of Ising spins:

For \( T < T_c \), there exists discontinuities in \( \partial F/\partial h \) when crossing the blue line (1st order PT line).
Magnetization

\[ M(T, h) = \frac{\partial F(T, h)}{\partial h} = \langle S_i \rangle. \]

At \( T < T_c \), \( M \) jumps as crossing \( h = 0 \) (1st order phase transition. )

At \( h = 0 \), naive Gibbsian measure \( \mu \) is not well defined

⇒ The system must be specified with the boundary condition at infinities:

\[ \mu(T, h = 0, a) = a\mu_+(T) + (1-a)\mu_-(T) \quad (0 < a < 1). \]

with \( \mu_\pm(T) = \mu(T, h \to \pm 0) \)
2nd order phase transition

At $T = T_c$, there exists no discontinuities on first derivatives of $F(T, h)$. What about second derivatives? ⇒ Magnetic susceptibility:

$$\chi = \frac{\partial^2 F}{\partial h^2} = \sum_i \langle S_0 S_i \rangle.$$

As $T \to T_c + 0$, the susceptibility diverges,

$$\chi \sim |(T - T_c)/T_c|^{-\gamma} \to \infty.$$
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Other scaling properties:

$$C := -(T - T_c) \frac{\partial^2 F}{\partial T^2} \sim |(T - T_c)/T_c|^{-\alpha} \to \infty,$$

$$M \sim |(T - T_c)/T_c|^\beta \to 0.$$ 

Scaling relation (Rushbrook identity):

$$\alpha + 2\beta + \gamma = 2.$$ 

(Only two of scaling exponents are independent!)
Scaling hypothesis

Assume that the free energy $G(T - T_c, h) = F(T, h)$ satisfies (Widom, 1965)

$$G(\lambda^s t, \lambda^r h) = G(t, h)/\lambda.$$ 

Taking derivatives of the both sides,

$$M(t, h) = \lambda^{r+1} M(\lambda^s t, \lambda^r h),$$
$$\chi(t, h) = \lambda^{2r+1} \chi(\lambda^s t, \lambda^r h),$$
$$C(t, h) = \lambda^{2s+1} C(\lambda^s t, \lambda^r h).$$

Putting $h = 0$ and $\lambda = 1/|t|^s$, we get

$$M(t, 0) \sim |t|^{-(r+1)/s}, \quad \chi(t, 0) \sim |t|^{-(2r+1)/s}, \quad C(t, 0) \sim |t|^{-(2s+1)/s}.$$ 

Then, $\alpha + 2\beta + \gamma = 2$ because

$$\alpha = \frac{2s + 1}{s}, \quad \beta = -\frac{r + 1}{s}, \quad \gamma = \frac{2r + 1}{s}.$$
Block spin transformation (I)

What justifies scaling hypothesis? \(\Rightarrow\) Renormalization Group

Block spin transformation (Kadanoff)

The number of total degrees of freedom: \(N \rightarrow N'.\) This defines the scaling factor

\[ \lambda = N'/N(< 1). \]

Idea: Long range feature is difficult to be computed from microscopic theory.

**Important!**

*Let’s perform coarse graining, and compute correlations between averaged spins!*
Block spin transformation (II)

We can generally denote the BST as

$$\mu_R(\{S_i^{(R)}\}) = \sum_{\{S_i\}} T(\{S_i^{(R)}\}, \{S_i\}) \mu(\{S_i\}).$$

$T$ is a transition probability, which satisfies

$$\sum_{\{S_i^{(R)}\}} T(\{S_i^{(R)}\}, \{S_i\}) = 1.$$

Assumption: $\mu_R$ is also Gibbsian with respect to block spins $S_i^{(R)}$:

$$\mu_R \propto \exp -H_R.$$

$$H_R(S^{(R)}) = R H(S) := -\ln \mu_R.$$

Free energy per unit cell: $F(H_R) = F(H)/\lambda$. 
Renormalization group flow

Operation $\mathcal{R}$ can be performed repeatedly on effective Hamiltonians $H$.

Fixed points $H^* = \mathcal{R}H^* \Rightarrow$ The system shows universal/self-similar behaviors (Wilson).

Important!

$RG$ provides a useful framework to extract and treat large-scale behaviors.
Functional renormalization group
General framework of FRG

Generating functional of connected Green functions:

$$\exp(W[J]) = \int \mathcal{D}\Phi \exp(-S[\Phi] + J \cdot \Phi).$$

infinite dimensional integration!

Possible remedy: Construct nonperturbative relations of Green functions!
(⇒ Functional techniques)

- Dyson-Schwinger equations
- 2PI formalism
- Functional renormalization group (FRG)
Wilsonian renormalization group

Classical action/Hamiltonian: \[ S[\phi] = \int d^d x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]. \]

“Block spins”: \[ \phi_p = \int d^d x e^{-ipx} \phi(x) \] for small momenta \( p \).

Field theoretic formulation of RG (Wilson, Kogut 1974):

\[ \exp -S_\Lambda[\phi] := N_\Lambda \int \prod_{|p| \geq \Lambda} d\phi_p \exp -S[\phi]. \]

Important!

Correlation functions \( \langle \phi_{p_1} \cdots \phi_{p_n} \rangle \) with low momenta \( |p_i| \leq \Lambda \) are calculable with \( S_\Lambda \) instead of the microscopic action \( S \).
Schwinger functional $W_k$ with an IR regulator $R_k$:

$$\exp(W_k[J]) = \int \mathcal{D}\phi \exp \left( -S[\phi] - \frac{1}{2} \phi \cdot R_k \cdot \phi + J \cdot \phi \right).$$

$R_k$: IR regulator, which controls low-energy excitations ($p^2 \leq k^2$).

$k$-derivatives of the both sides:

$$\partial_k \exp(W_k[J]) = \int \mathcal{D}\phi - \frac{1}{2} \phi R_k \phi \exp \left( -S[\phi] - \frac{1}{2} \phi R_k \phi + J \phi \right)$$

$$= -\frac{1}{2} \frac{\delta}{\delta J} R_k \frac{\delta}{\delta J} \exp W_k[J]$$

Flow equation

$$\partial_k W_k = -\frac{1}{2} \frac{\delta W_k}{\delta J} R_k \frac{\delta W_k}{\delta J} - \frac{1}{2} R_k \frac{\delta^2 W_k}{\delta J \delta J}.$$
The 1PI effective action $\Gamma_k$ is introduced via the Legendre trans.:

$$\Gamma_k[\varphi] + \frac{1}{2} \varphi \cdot R_k \cdot \varphi = J[\varphi] \cdot \varphi - W_k[J[\varphi]],$$

which obeys the flow equation (Wetterich 1993, Ellwanger 1994, Morris 1994)

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \frac{\partial_k R_k}{\delta^2 \Gamma_k[\Phi]/\delta \Phi \delta \Phi + R_k} = \partial_k R_k$$

Properties of $\Gamma_k$: $\Gamma_k \to S$ as $R_k \to \infty$, and $\Gamma_k \to \Gamma$ as $R_k \to 0$.

**Important!**

*Functional implementation of “block spin transformations” keeps all the information of microscopic systems.*
Generalized flow equation of FRG

\[ \delta S_k[\Phi]: \text{Some function of } \Phi \text{ with a parameter } k. \quad \text{(IR regulator)} \]

\[ k\text{-dependent Schwinger functional} \]

\[ \exp(W_k[J]) = \int \mathcal{D}\Phi \exp\left[ - (S[\Phi] + \delta S_k[\Phi]) + J \cdot \Phi \right] \]

Flow equation

\[ -\partial_k W_k[J] = \langle \partial_k \delta S_k[\Phi] \rangle_J \]

\[ = \exp(-W_k[J]) \partial_k (\delta S_k) \left[ \delta/\delta J \right] \exp(W_k[J]) \]

Consequence

We get a (functional) differential equation instead of a (functional) integration!
Choice of IR regulators $\delta S_k$ is arbitrary.

Optimization:
Choose the “best” IR regulator, which validates systematic truncation of an approximation scheme.

Optimization criterion (Litim 2000, Pawlowski 2007):
- IR regulators $\delta S_k$ make the system gapped by a typical energy $k^2/2m$ of the parameter $k$.
- High-energy excitations ($\gtrsim k^2/2m$) should decouple from the flow of FRG at the scale $k$.
- Choose $\delta S_k$ stabilizing calculations and making it easier.
Conventional approach: Wetterich equation

At high energies, perturbation theory often works well.  
⇒ Original fields control physical degrees of freedom.

IR regulator for bare propagators (\(\sim\) mass term):  
\[
\delta S_k[\Phi] = \frac{1}{2} \Phi_\alpha R^\alpha_\beta \Phi_\beta.
\]

Flow equation of 1PI effective action \(\Gamma_k[\Phi]\) (Wetterich 1993)

\[
\partial_k \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \frac{\partial_k R_k}{\delta^2 \Gamma_k[\Phi]/\delta \Phi \delta \Phi + R_k}
\]
In the infrared region, collective bosonic excitations emerge quite in common. (e.g.) Another low-energy excitation emerges in the $\Phi\Phi$ channel

Vertex IR regulator: $\delta S_k = \frac{1}{4!} g_k^{\alpha\beta\gamma\delta} \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta$. 

Flow equation with the vertex IR regulator (YT, PTEP2014, 023A04) 

$$ \partial_k \Gamma_k[\Phi] = \begin{array}{c} \text{tree} \end{array} + \begin{array}{c} \text{loop} \end{array} + \begin{array}{c} \text{two-loop} \end{array} + \begin{array}{c} \text{three-loop} \end{array} + \begin{array}{c} \text{four-loop} \end{array} + \begin{array}{c} \text{five-loop} \end{array} $$
Application of fermionic FRG to the BCS-BEC crossover
Cold atomic physics

Ultracold fermions provides examples of strongly-correlated fermions. High controllability can tune effective couplings with real experiments!

(Typically, $T \sim 100 \text{nK}$, and $n \sim 10^{11-14} \ \text{cm}^{-3}$)
BCS-BEC crossover

EFT: Two-component fermions with an attractive contact interaction.

\[ S = \int d^4 x \left[ \bar{\psi}(x) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(x) + g \bar{\psi}_1(x) \bar{\psi}_2(x) \psi_2(x) \psi_1(x) \right] \]

Question

Is it possible to treat EFT systematically to describe the BCS-BEC crossover?
General strategy

We will calculate $T_c/\varepsilon_F$ and $\mu/\varepsilon_F$.

$\Rightarrow$ Critical temperature and the number density must be calculated.

We expand the 1PI effective action in the symmetric phase:

$$\Gamma_k[\overline{\psi}, \psi] = \beta F_k(\beta, \mu) + \int_p \overline{\psi}_p [G^{-1}(p) - \Sigma_k(p)] \psi_p$$

$$+ \int_{p,q,q'} \Gamma^{(4)}_k(p) \overline{\psi}_{\uparrow,\frac{p}{2}+q} \overline{\psi}_{\downarrow,\frac{p}{2}-q} \psi_{\downarrow,\frac{p}{2}-q'} \psi_{\uparrow,\frac{p}{2}+q'}.$$ 

Critical temperature and the number density are determined by

$$\frac{1}{\Gamma_0^{(4)}(p = 0)} = 0, \quad n = \int_p \frac{-2}{G^{-1}(p) - \Sigma_0(p)}.$$
Case 1  Negative scattering length $(k_F a_s)^{-1} \ll -1$.
⇒ Fermi surface exists, and low-energy excitations are fermionic quasi-particles.

Shanker's RG for Fermi liquid (Shanker 1994)
Functional implementation of Shanker’s RG

RG must keep low-energy fermionic excitations under control.

\[ \delta S_k = \int_p \bar{\psi}_p R_k^{(f)}(p) \psi_p \]  

with

\[ R_k^{(f)}(p) = \text{sgn}(\xi(p)) \left( \frac{k^2}{2m} - |\xi(p)| \right) \theta \left( \frac{k^2}{2m} - |\xi(p)| \right) \]

Flow equation of the self-energy \( \Sigma_k \) and the four-point 1PI vertex \( \Gamma_k^{(4)} \):

\[ \partial_k \sqrt{ } = \quad \partial_k \quad = \quad \text{square} + \text{triangle} \]
Flow of fermionic FRG: effective four-fermion interaction

- Particle-particle loop ⇒ RPA & BCS theory
- Particle-hole loop gives screening of the effective coupling at \( k \sim k_F \)

\[
T_c^{\text{BCS}} = \varepsilon_F \frac{8 e^{\gamma E - 2}}{\pi} e^{-\pi / 2k_F |a_s|} \Rightarrow T_c^{\text{BCS}} / 2.2. \quad \text{(Gorkov, Melik-Barkhudarov, 1961)}
\]
Flow of fermionic FRG: self-energy

Local approximation on self-energy: $\Sigma_k(p) \simeq \sigma_k$.

- High energy: $\sigma_k \simeq (\text{effective coupling}) \times (\text{number density}) \sim 1/k$
- Low energy: $\partial_k \sigma_k \sim 0$ due to the particle-hole symmetry.
Transition temperature and chemical potential in the BCS side

Critical temperature $T_c/\varepsilon_F$ is significantly reduced by a factor 2.2 in $(k_F a_s)^{-1} \lesssim -1$, and the self-energy effect on it is small in this region.

$\mu(T_c)/\varepsilon_F$ is largely changed from 1 even when $(k_F a_s)^{-1} \lesssim -1$. 

BEC side

Case 2 Positive scattering length : \((k_F a_s)^{-1} \gg 1\)
⇒ Low-energy excitations are one-particle excitations of composite dimers.

\[ E_b = \frac{1/a_s^2}{2m_r} \]
BEC side

Case 2 Positive scattering length : \((k_F a_s)^{-1} \gg 1\)
⇒ Low-energy excitations are one-particle excitations of composite dimers.

Several approaches for describing BEC of composite bosons. (Pros/Cons)

- Auxiliary field method
  (Easy treatment within MFA/ Fierz ambiguity in their introduction)
- Fermionic FRG (\(\Leftarrow\) We develop this method!)
  (Unbiased and unambiguous/ Nonperturbative treatment is necessary)
Vertex IR regulator & Flow equation

Optimization can be satisfied with the vertex IR regulator:

\[
\delta S_k = \int \frac{g^2 R_k^{(b)}(p)}{1 - g R_k^{(b)}(p)} \int \bar{\psi}_{\uparrow, \frac{p}{2} + q} \psi_{\downarrow, \frac{p}{2} - q} \psi_{\downarrow, \frac{p}{2} - q'} \psi_{\uparrow, \frac{p}{2} + q'}
\]

Flow equation up to fourth order (YT, PTEP 2014 023A04, YT, arXiv:1402.0283):

\[
\partial_k = \Gamma_k^{(4)}(p) = \frac{1}{\Gamma_k^{(4)}(p)} - m^2 a_s \frac{8 \pi}{\Gamma_k^{(4)}(p)} \left( i p^0 + \frac{p^2}{4m} \right) - R_k^{(b)}(p)
\]
Flow of fermionic FRG: self-energy

Flow equation of the self-energy:

$$\partial_k \Sigma_k(p) = \int_l \frac{\partial_k \Gamma_k^{(4)}(p + l)}{i l^0 + l^2/2m + 1/2ma_s^2 - \Sigma_k(l)}.$$ 

If $|\Sigma_k(p)| \ll 1/2ma_s^2$,

$$\Sigma_k(p) \approx \int_l \frac{\Gamma_k^{(4)}(p + l)}{i l^0 + l^2/2m + 1/2ma_s^2}.$$ 

$$\approx \int \frac{d^3q}{(2\pi)^3} \frac{(8\pi/m^2a_s)n_B(q^2/4m + m^2a_s/8\pi R_k^{(b)}(q))}{ip^0 + q^2/4m + m^2a_s/8\pi R_k^{(b)}(q) - (q+p)^2/2m - 1/2ma_s^2}.$$ 

Estimate of $|\Sigma_k(p)|$:

$$|\Sigma_k(p)| \lesssim \frac{1}{2ma_s^2} \times (\sqrt{2mT}a_s)^3 \times n_B(k^2/4m).$$ 

$\Rightarrow$ Our approximation is valid up to $(k^2/2m)/T \sim (k_Fa_s)^3 \ll 1$. 
Critical temperature in the BEC side

Number density:

\[
n = \int_p \frac{-2}{i p^0 + p^2/2m + 1/2ma_s^2 - \Sigma_0(p)}
\]

\[
\approx \frac{(2mT_c)^{3/2}}{\pi^2} \sqrt{\frac{\pi}{2}} \zeta(3/2).
\]

Critical temperature and chemical potential:

\[
T_c/\varepsilon_F = 0.218, \quad \mu/\varepsilon_F = -1/(k_Fa_s)^2.
\]

\[\Rightarrow\] Transition temperature of BEC.

Consequence

FRG with vertex regulator provides a nonperturbative description of many-body composite particles.
We discuss the whole region of the BCS-BEC crossover with fermionic FRG.
⇒ Combine two different formalisms appropriate for BCS and BEC sides.

Minimal set of the flow equation for $\Sigma_k$ and $\Gamma_k^{(4)}$:

\[
\partial_k \rightarrow \quad = \quad + \quad \rightarrow \quad
\]

\[
\partial_k \quad = \quad \rightarrow \quad + \quad \rightarrow \quad
\]
Flow of fermionic FRG with multiple regulators

Flow of four-point vertex:
Important property: fermions decouple from RG flow at the low energy region.
- In BCS side, fermions decouples due to Matsubara freq. \( (k^2/2m \lesssim \pi T) \).
- In BEC side, fermions decouples due to binding E. \( (k^2/2m \lesssim 1/2ma_s^2) \).

Approximation on the flow of the four-point vertex at low energy:

\[
\partial_k \simeq \begin{array}{c}
\text{flow of self-energy:} \\
\text{At a low-energy region, the above approx. gives}
\end{array}
\]

\[
\partial_k \rightarrow \begin{array}{c}
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\text{At a low-energy region, the above approx. gives}
\end{array}
\]

\[
\partial_k \approx \partial_k
\]
Qualitative behaviors of the BCS-BEC crossover from f-FRG

Approximations on the flow equation have physical interpretations.

Four-point vertex: Particle-particle RPA. The Thouless criterion

\(1/\Gamma^{(4)}(p = 0) = 0\) gives

\[
\frac{1}{a_s} = -\frac{2}{\pi} \int_{0}^{\infty} \sqrt{2m\varepsilon} d\varepsilon \left[ \frac{\tanh \frac{\beta}{2} (\varepsilon - \mu)}{2(\varepsilon - \mu)} - \frac{1}{2\varepsilon} \right]
\]

⇒ BCS gap equation at \(T = T_c\).

Number density: \(n = -2 \int 1/(G^{-1} - \Sigma)\).

\[
n = -2 \int_{p}^{(T)} G(p) - \frac{\partial}{\partial \mu} \int_{p}^{(T)} \ln \left[ 1 + \frac{4\pi a_s}{m} \left( \Pi(p) - \frac{m\Lambda}{2\pi^2} \right) \right].
\]

⇒ Pairing fluctuations are taken into account. (Nozieres, Schmitt-Rink, 1985)

Consequence

*We established the fermionic FRG which describes the BCS-BEC crossover.*
Summary
RG provides a useful framework to extract and treat large-scale behaviors.

Functional implementation of coarse graining provides systematic treatment of field theories.

Fermionic FRG is a promising formalism for interacting fermions.
⇒ Separation of energy scales can be realized by optimization.
⇒ Very flexible form for various approximation schemes.

Fermionic FRG is applied to the BCS-BEC crossover.
⇒ BCS side: GMB correction + the shift of Fermi energy from $\mu$.
⇒ BEC side: BEC without explicit bosonic fields.
⇒ whole region: Crossover physics is successfully described at the quantitative level with a minimal setup on f-FRG.