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Expanding Polyhedral Universe in Regge Calculus

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2. Regge Calculus における Hilbert 作用
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5. Summary

1. 单体分割

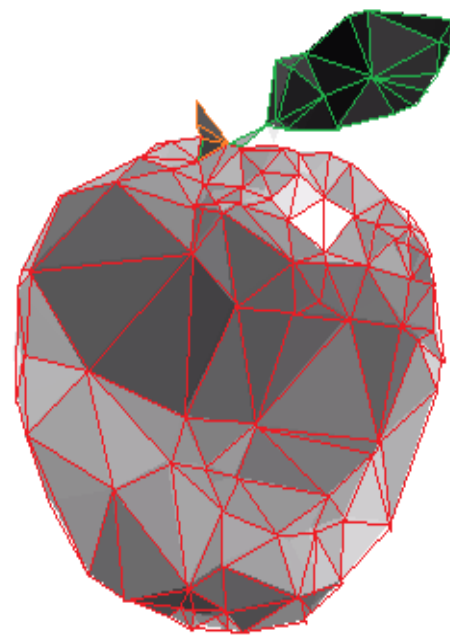
2次元曲面の三角分割



滑らかな曲面

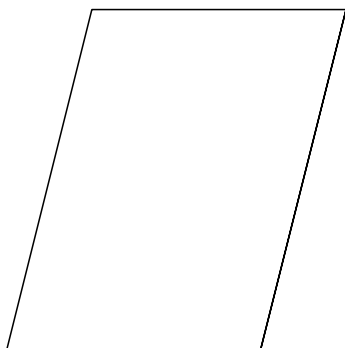
2次元曲面の三角分割

三角形で近似

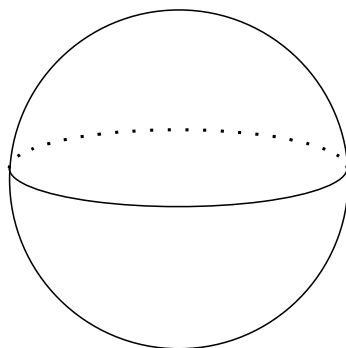


曲率... 三角形の頂点に定義

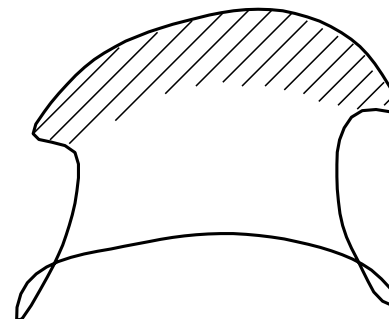
平面 (曲率ゼロ)



球面 (正曲率)

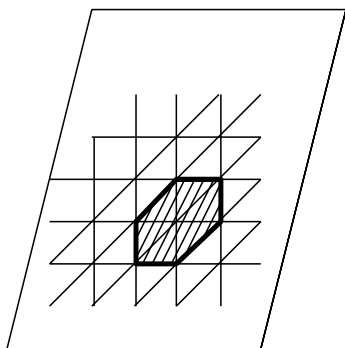


鞍状面 (負曲率)

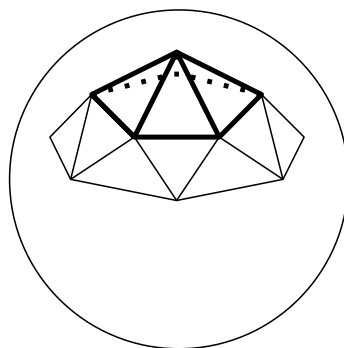


曲率・・・三角形の頂点に定義

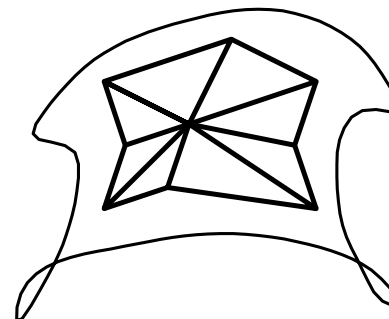
平面（曲率ゼロ）



球面（正曲率）

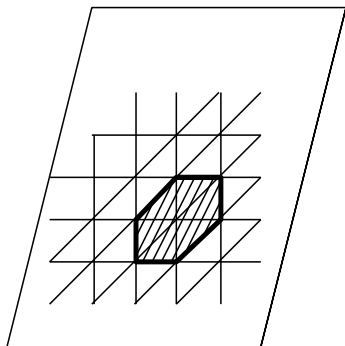


鞍状面（負曲率）

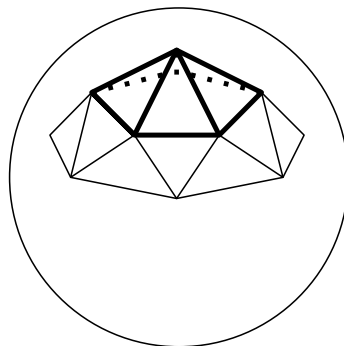


曲率... 三角形の頂点に定義

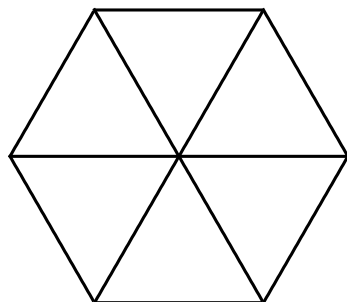
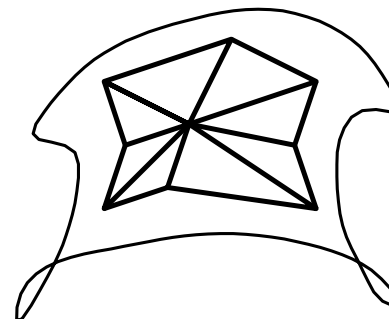
平面 (曲率ゼロ)



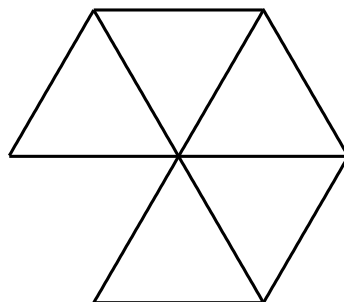
球面 (正曲率)



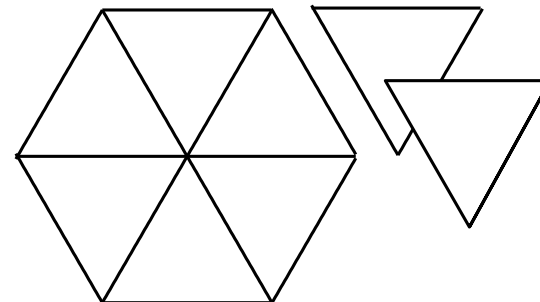
鞍状面 (負曲率)



過不足なし



不足



過剰

n -単体 \cdots $n + 1$ 個の頂点を持つ n 次元凸面体

0-単体



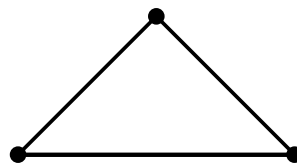
点

1-単体



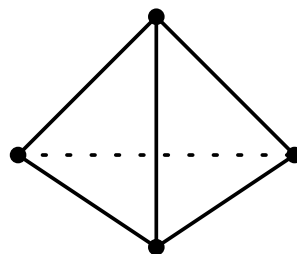
線分

2-単体



三角形

3-単体



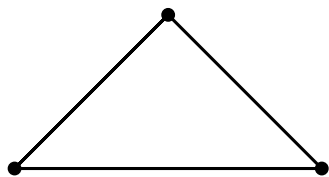
四面体

⋮

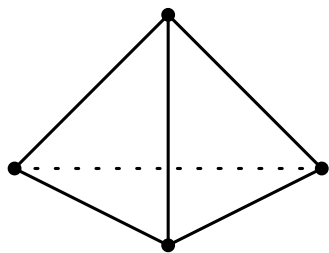


⋮

単体の構造

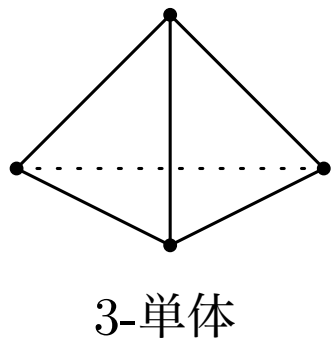
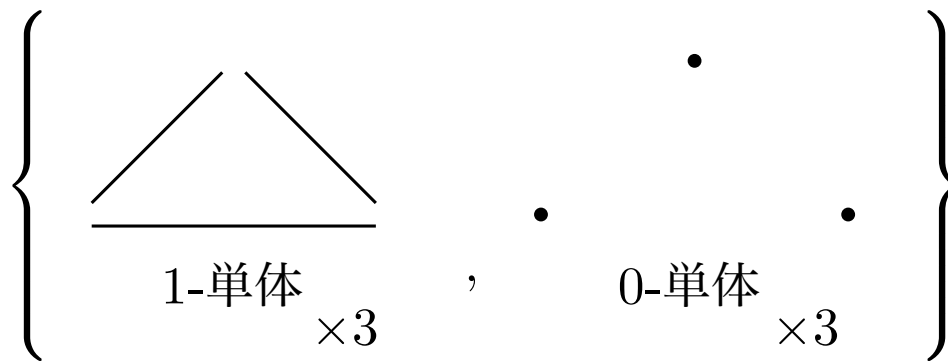
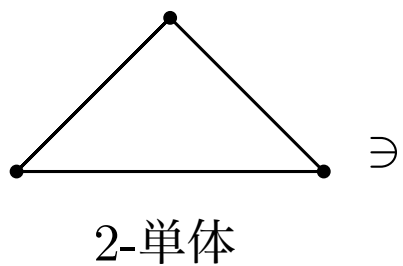


2-単体

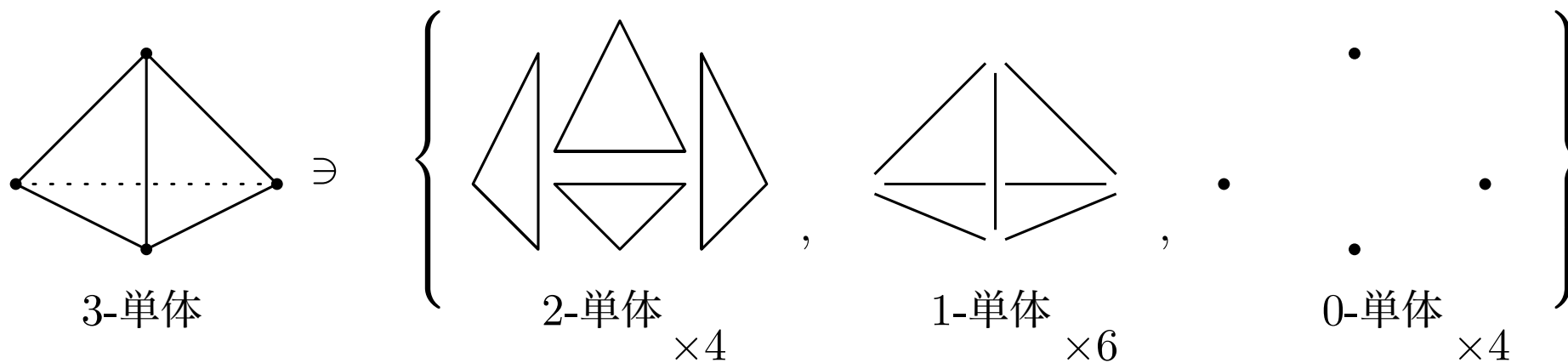
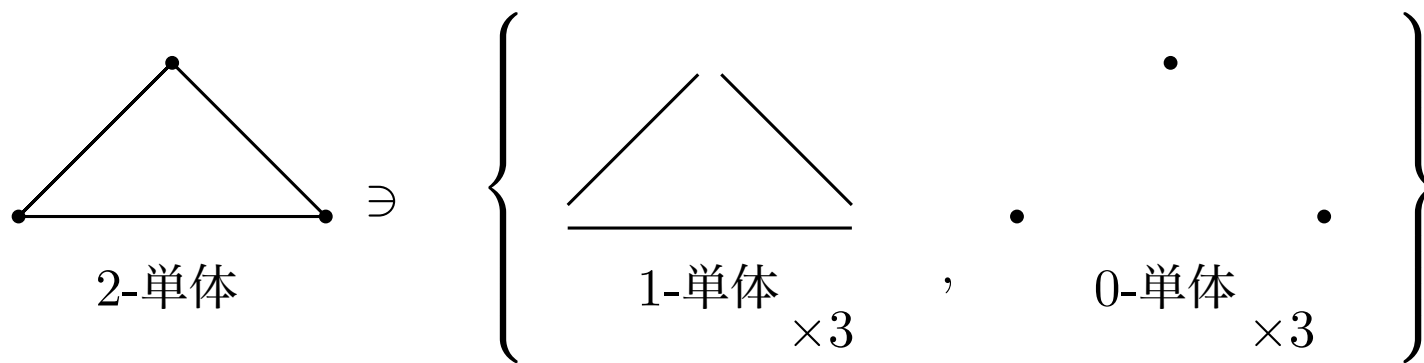


3-単体

単体の構造



単体の構造



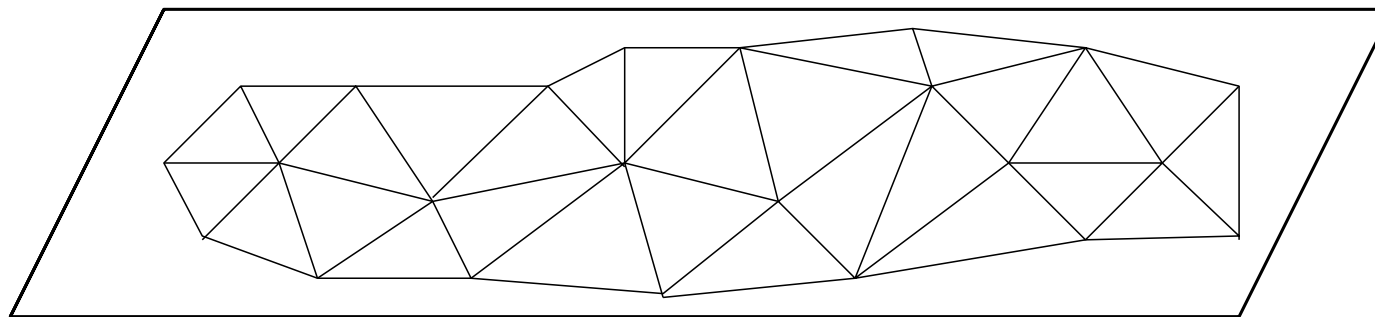
単体の構造

Volume	:	A_b	A_h	
名称	:	Boundary	Hinge: h	
n -単体	\ni	$\{ (n-1)\text{-単体} \times_{n+1} C_1, (n-2)\text{-単体} \times_{n+1} C_2, \dots, 0\text{-単体} \times_{n+1} C_n \}$		

Delaunay 格子と Voronoi ポリゴン

単体分割によって作られた格子

... Delaunay 格子



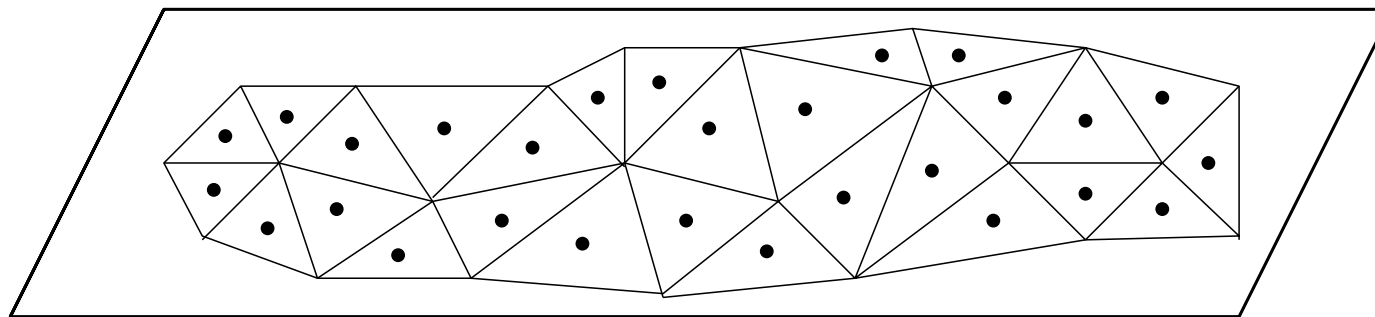
1. 各単体の外心を求める
2. 隣り合う外心を結ぶ

... Voronoi ポリゴン

Delaunay 格子と Voronoi ポリゴン

単体分割によって作られた格子

... Delaunay 格子



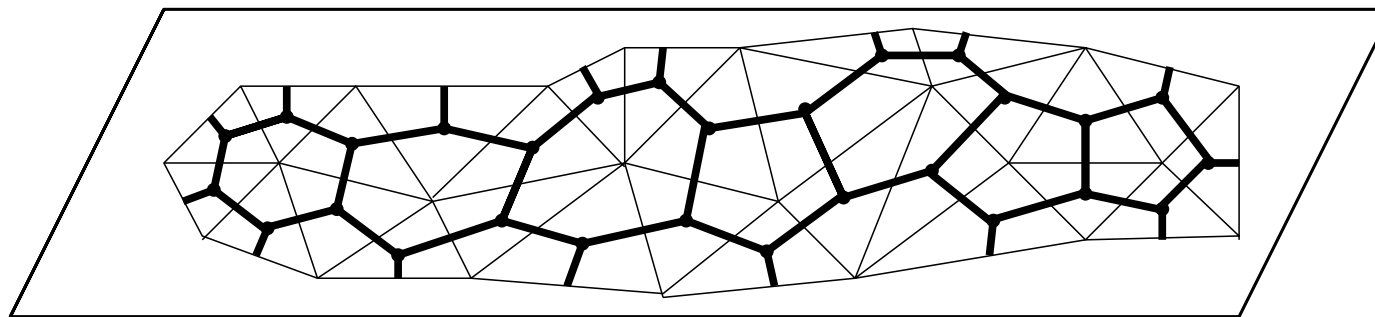
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Delaunay 格子と Voronoi ポリゴン

単体分割によって作られた格子

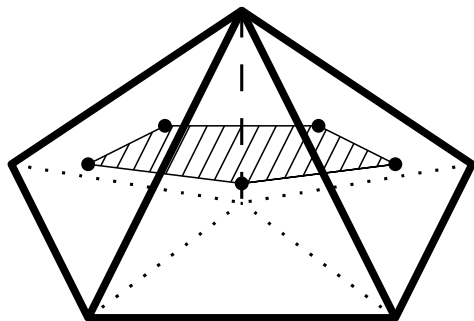
... Delaunay 格子



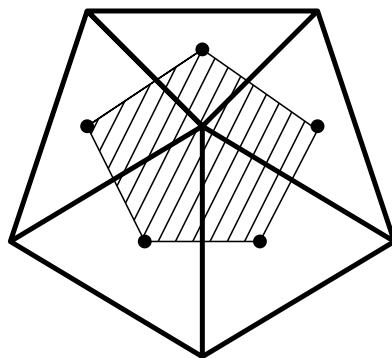
1. 各単体の外心を求める
2. 隣り合う外心を結ぶ

... Voronoi ポリゴン

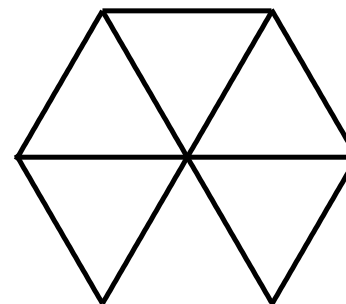
Hinge と Voronoi ポリゴン と Deficit Angle



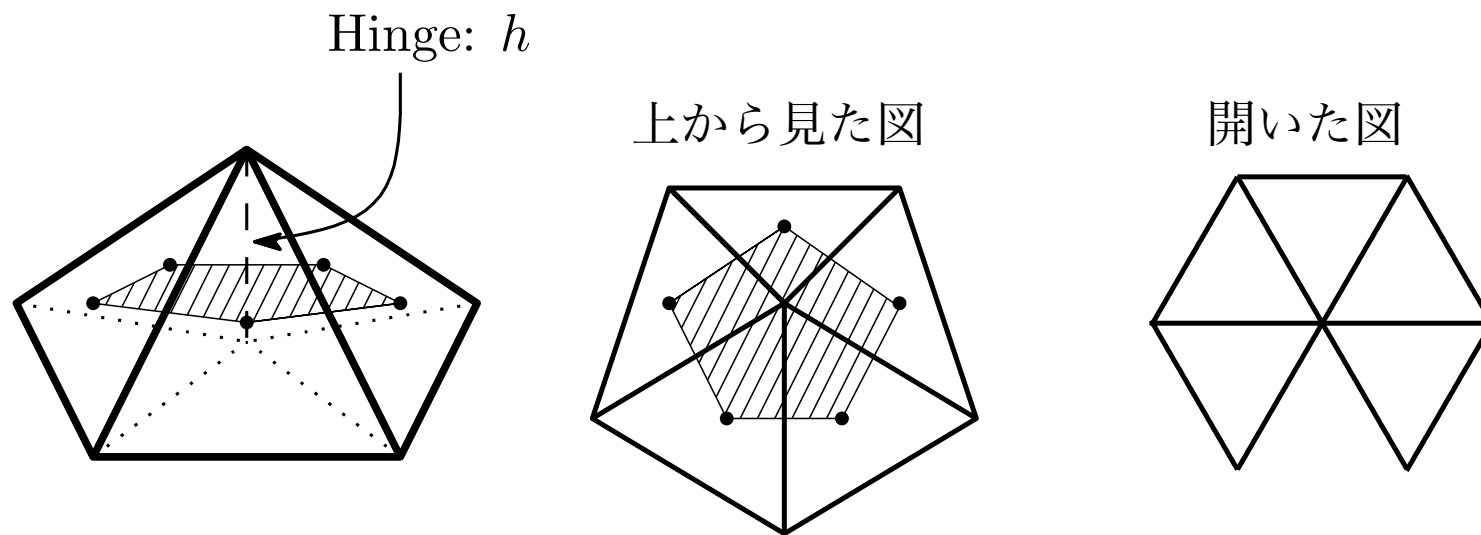
上から見た図



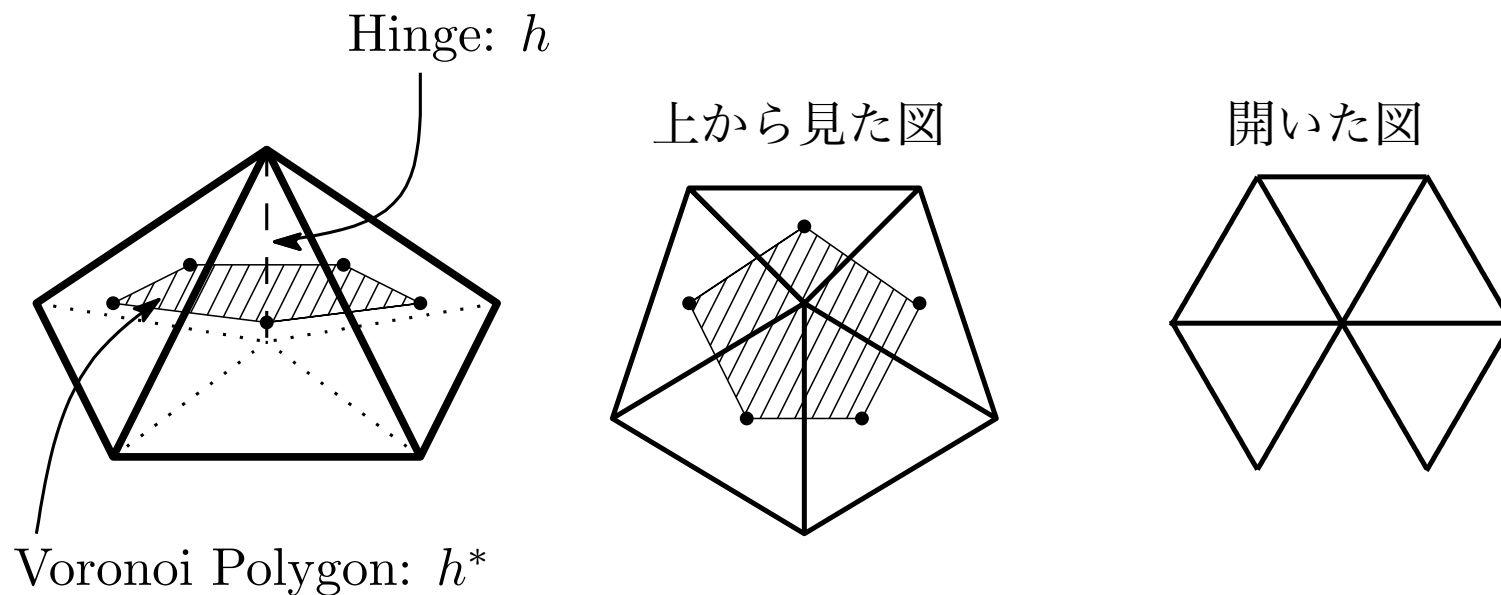
開いた図



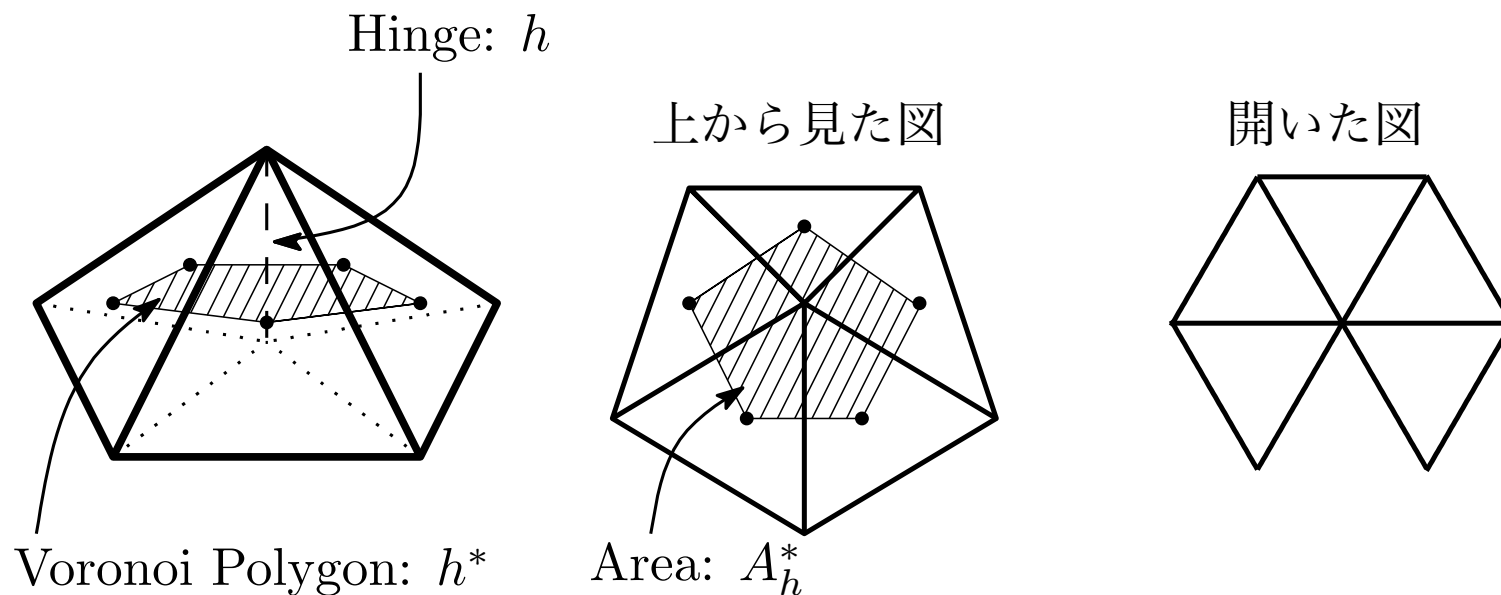
Hinge と Voronoi ポリゴン と Deficit Angle



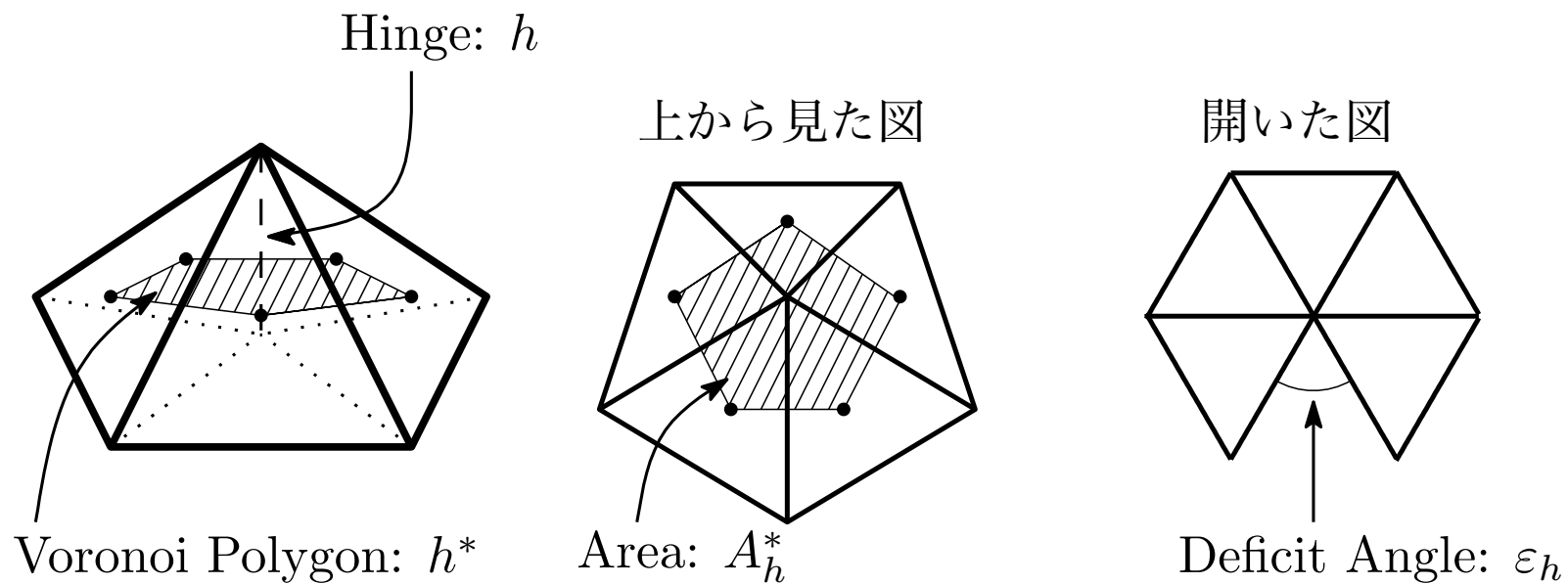
Hinge と Voronoi ポリゴンと Deficit Angle



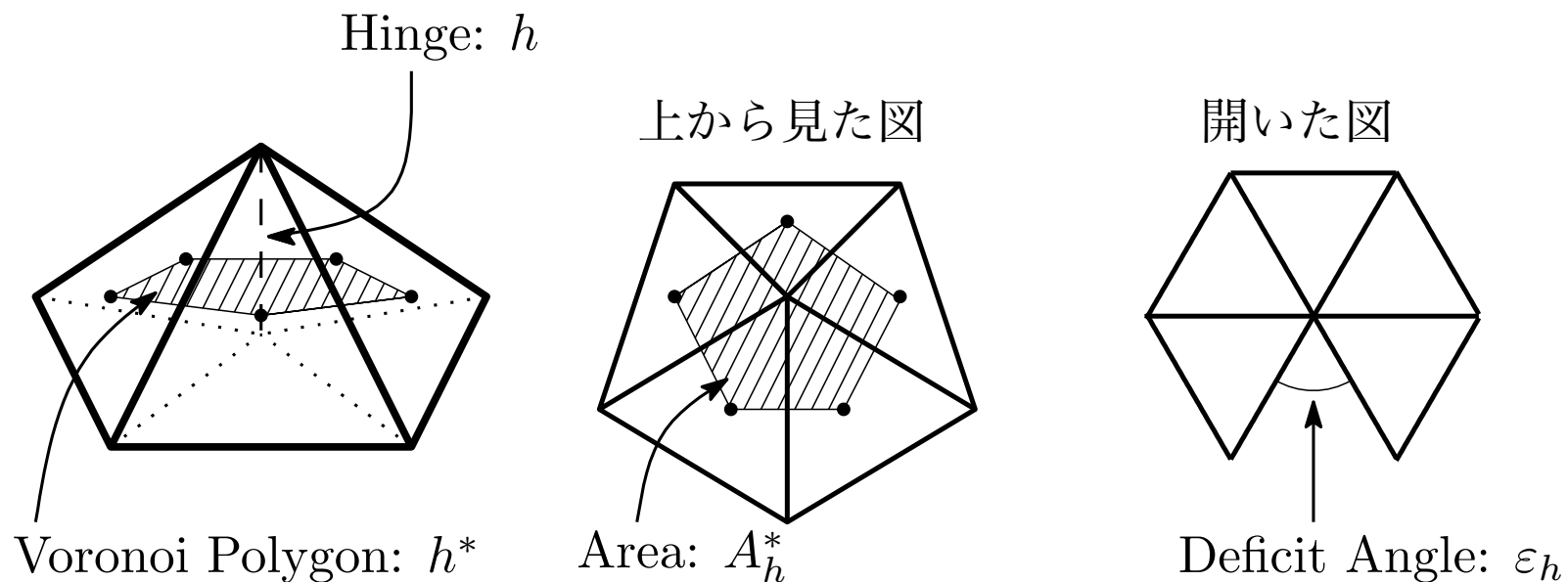
Hinge と Voronoi ポリゴン と Deficit Angle



Hinge と Voronoi ポリゴン と Deficit Angle



Hinge と Voronoi ポリゴン と Deficit Angle



h と h^* と ε_h : 1対1対1対応

Regge Calculus とは (T. Regge, Il Nouvo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる
2. 単体の配置を変化させる

Regge Calculus とは (T. Regge, Il Nouvo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる
→ Regge Calculus
2. 単体の配置を変化させる

Regge Calculus とは (T. Regge, Il Nouvo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる

→ Regge Calculus

2. 単体の配置を変化させる

→ Dynamical Triangulation

Regge Calculus とは (T. Regge, Il Nouvo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる

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2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Grav. **14**, 199 (1997))

- **Ricci Scalar Curvature**
- Proper Volume Element
- Hilbert Action

Ricci Scalar Curvature の変形

$$R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$$

Ricci Scalar Curvature の変形

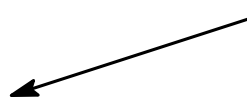
$$\underline{\underline{R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}}}$$

$$R = D(D - 1)K$$

Ricci Scalar Curvature の変形

$$\underline{\underline{R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}}}$$

Dimension

$$R = D (D - 1) K$$


Ricci Scalar Curvature の変形

$$\underline{\underline{R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}}}$$

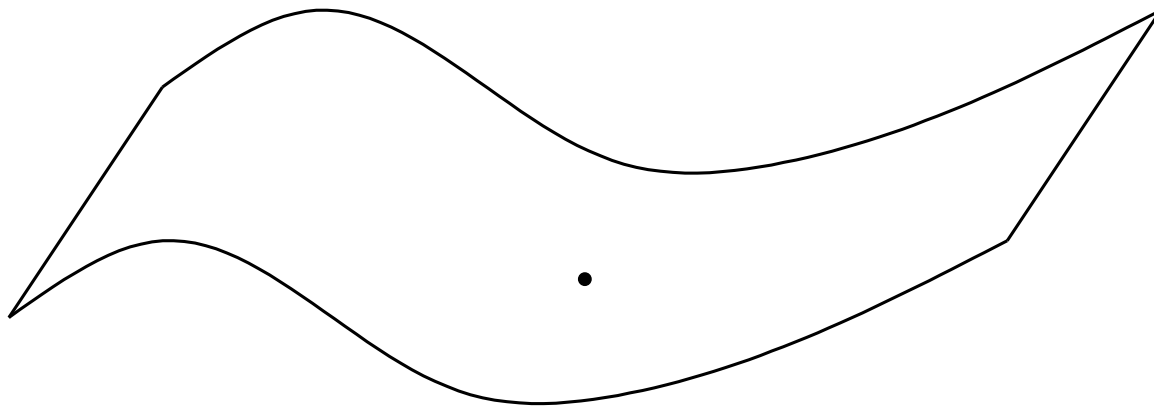
Dimension

$$R = D(D - 1)K$$

Gaussian Curvature

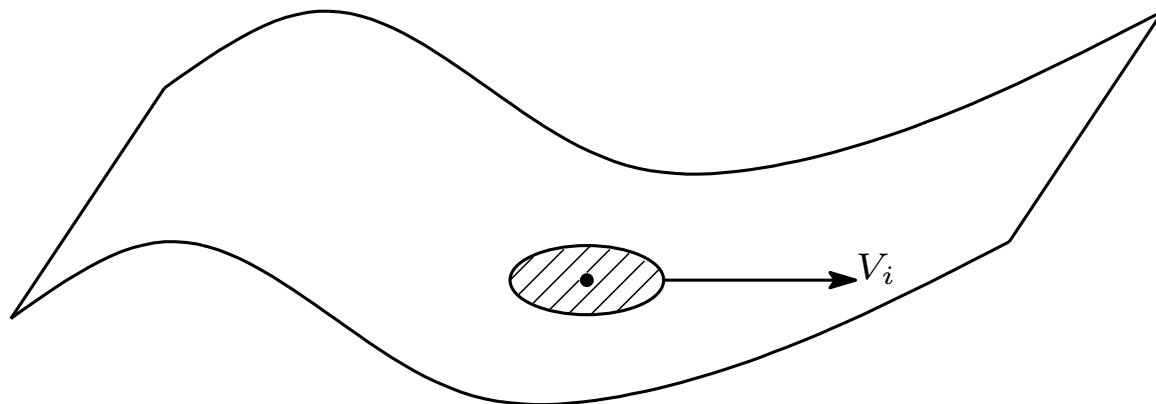
Gaussian Curvature

K



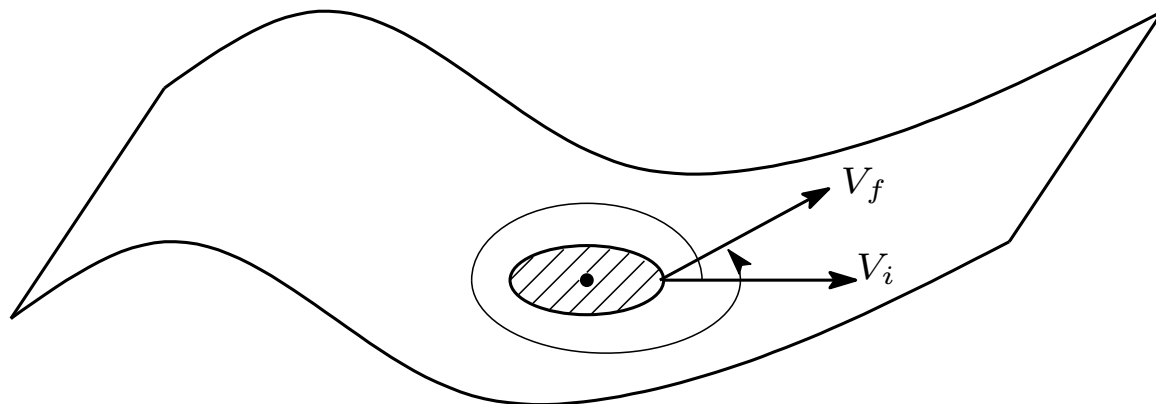
Gaussian Curvature

K



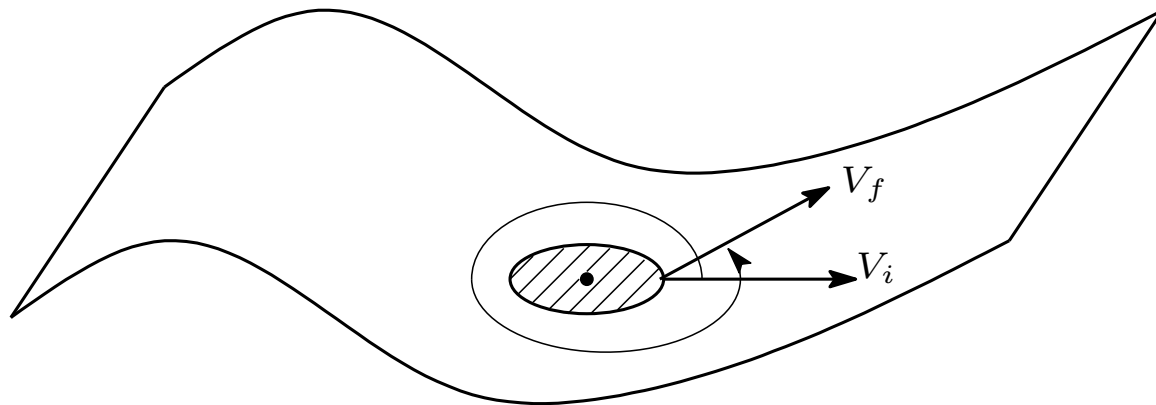
Gaussian Curvature

K



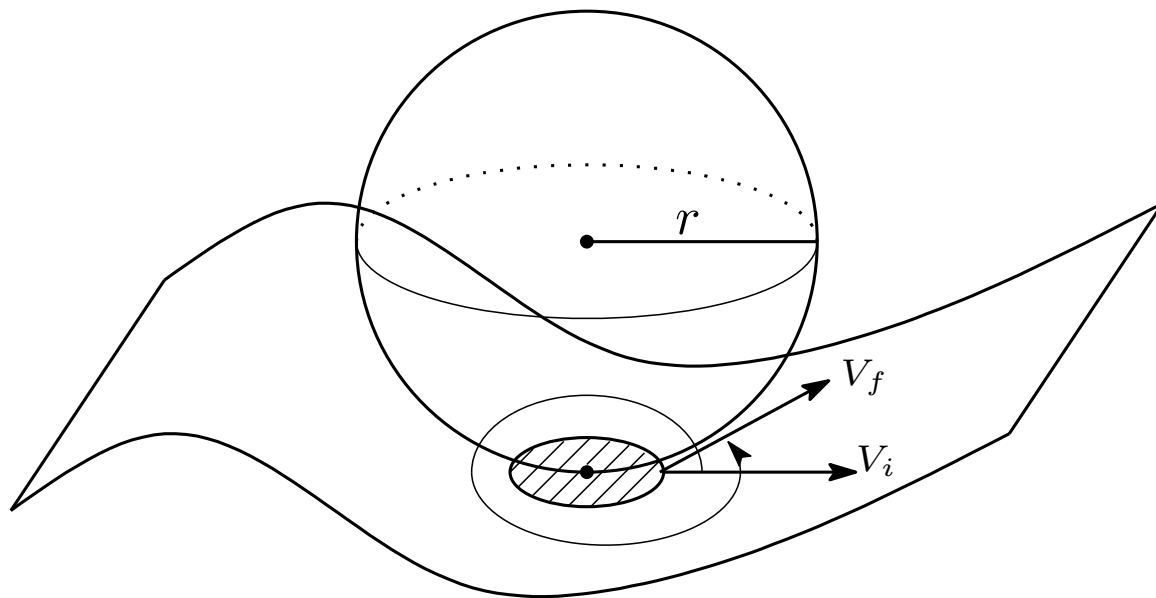
Gaussian Curvature

$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}}$$



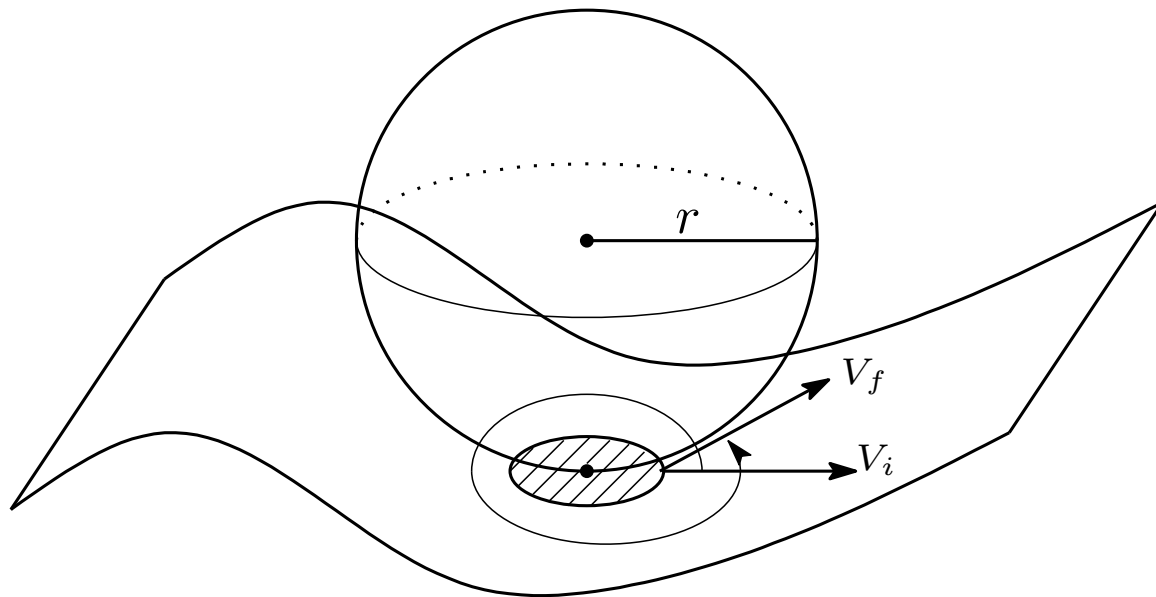
Gaussian Curvature

$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}}$$

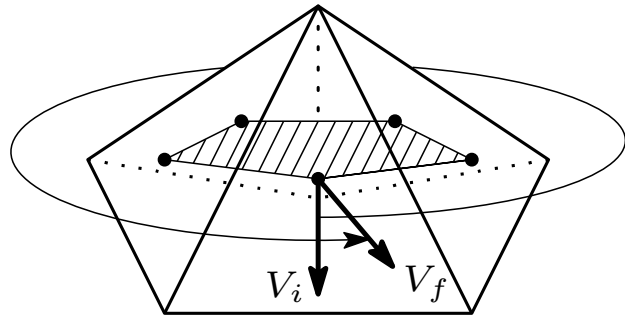


Gaussian Curvature

$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}} = \frac{1}{r^2}$$

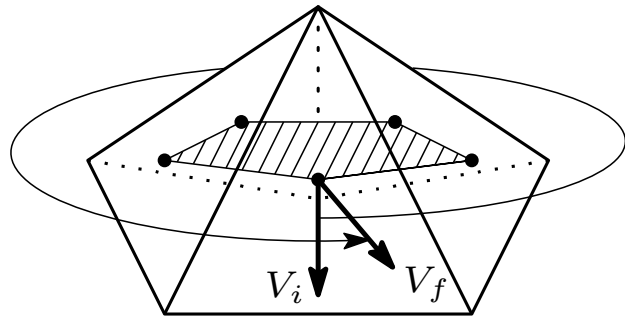


Gaussian Curvature

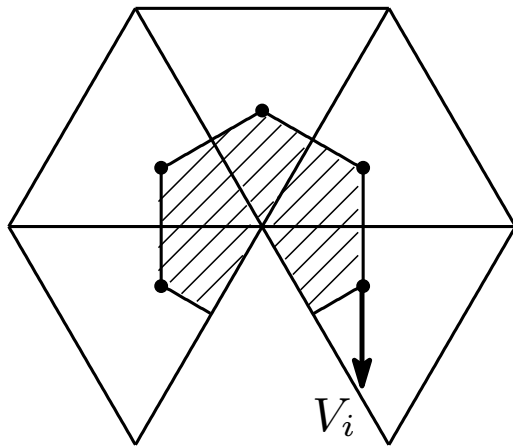


Area Circumnavigated
 $= A_h^*$

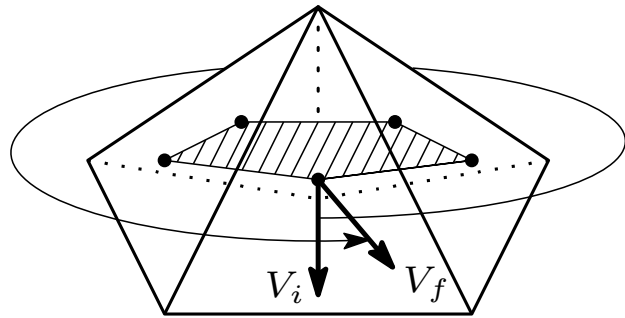
Gaussian Curvature



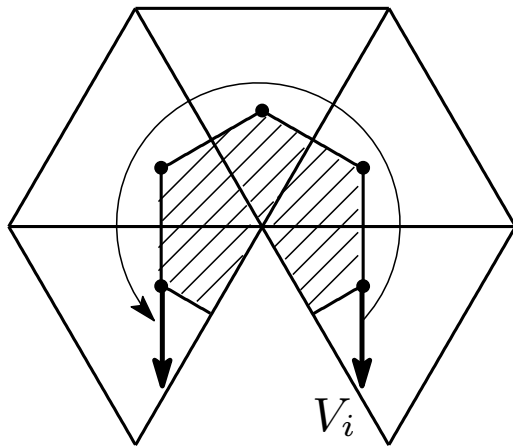
$$\text{Area Circumnavigated} \\ = A_h^*$$



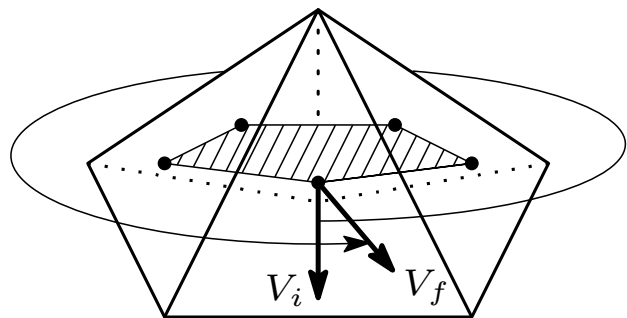
Gaussian Curvature



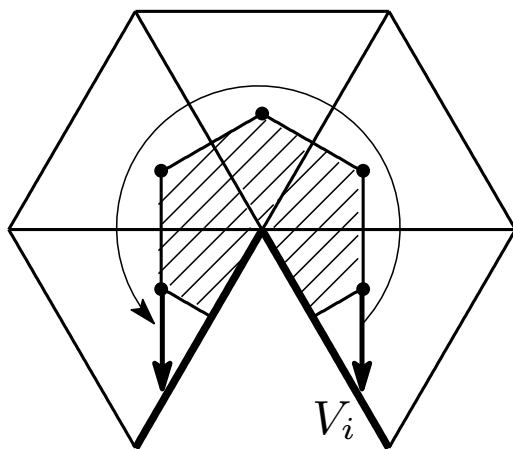
$$\text{Area Circumnavigated} \\ = A_h^*$$



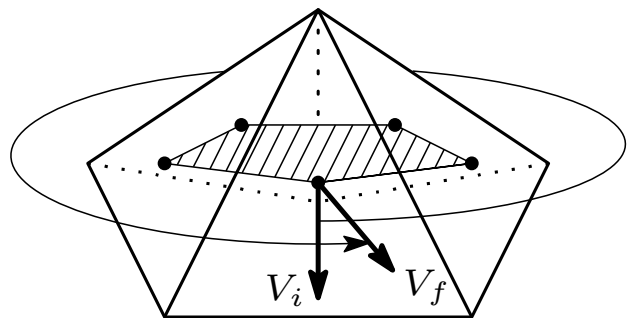
Gaussian Curvature



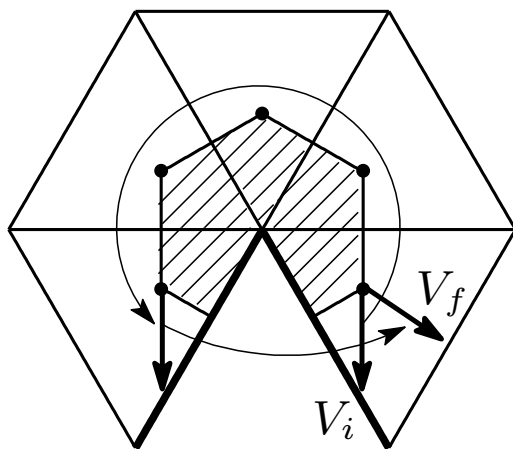
Area Circumnavigated
 $= A_h^*$



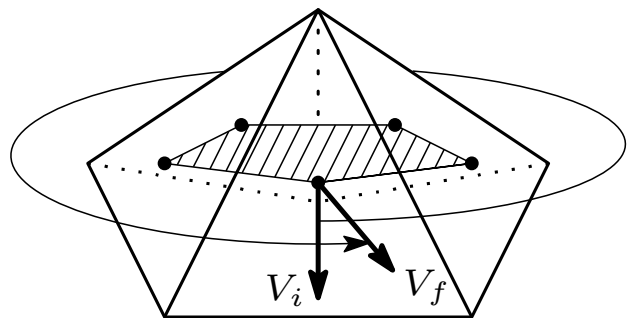
Gaussian Curvature



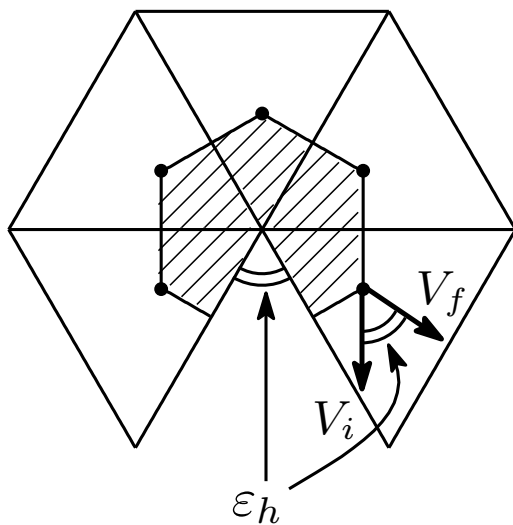
$$\text{Area Circumnavigated} \\ = A_h^*$$



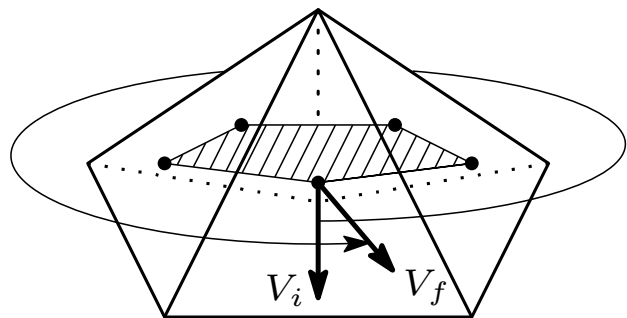
Gaussian Curvature



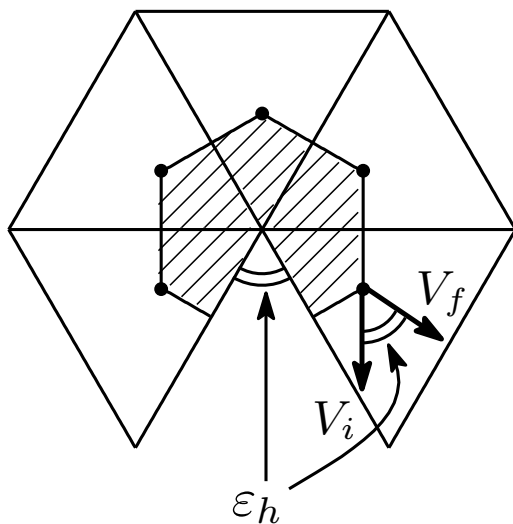
$$\text{Area Circumnavigated} \\ = A_h^*$$



Gaussian Curvature



Area Circumnavigated
 $= A_h^*$



Angle that Vector
is Rotated
 $= \epsilon_h$

Ricci Scalar Curvature in Regge Calculus

$$K_{\text{Regge}} = \frac{\varepsilon_h}{A_h^*} \left(\longrightarrow \frac{1}{r^2} \right)$$

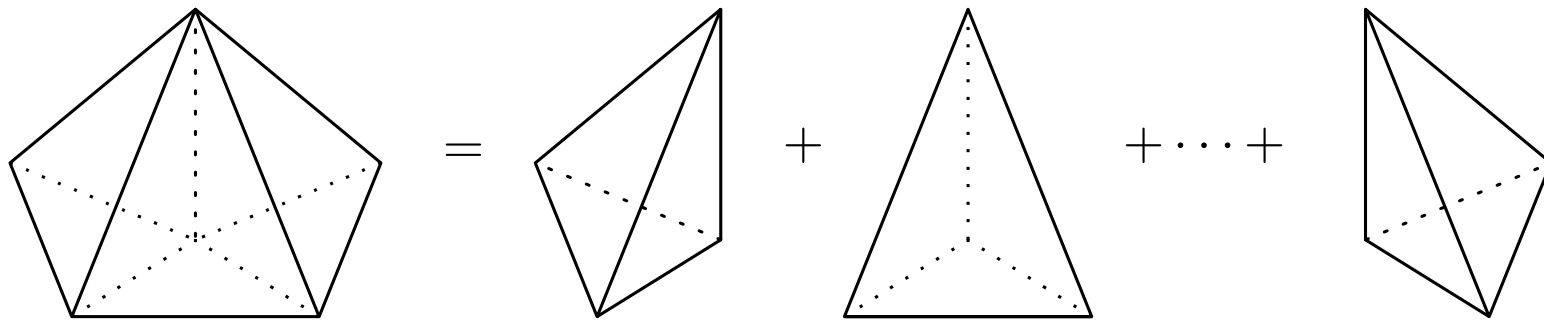
$$R_{\text{Regge}} = D(D-1) \frac{\varepsilon_h}{A_h^*}$$

2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Grav. **14**, 199 (1997))

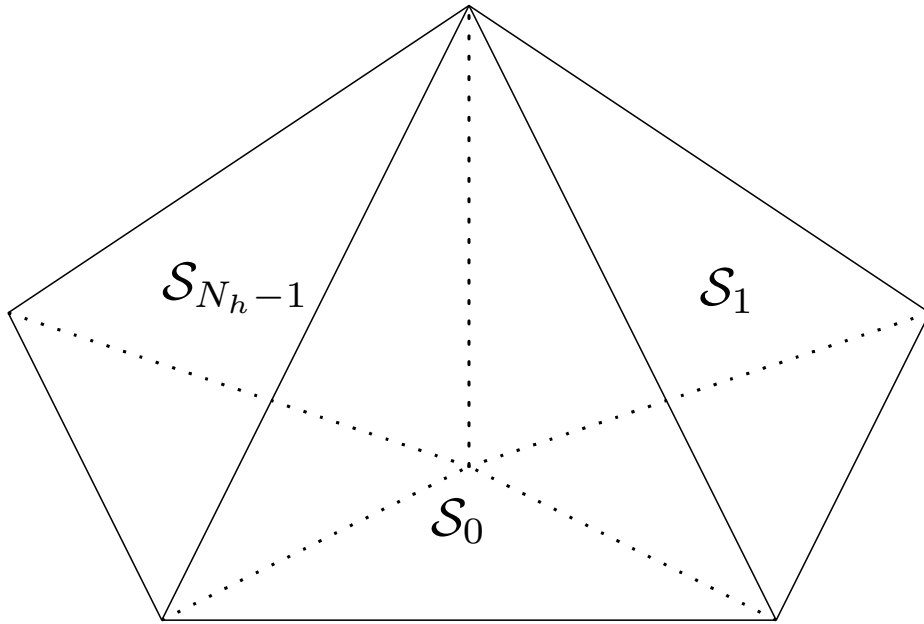
- Ricci Scalar Curvature
- **Proper Volume Element**
- Hilbert Action

複体 ... 有限個の単体からなる立体



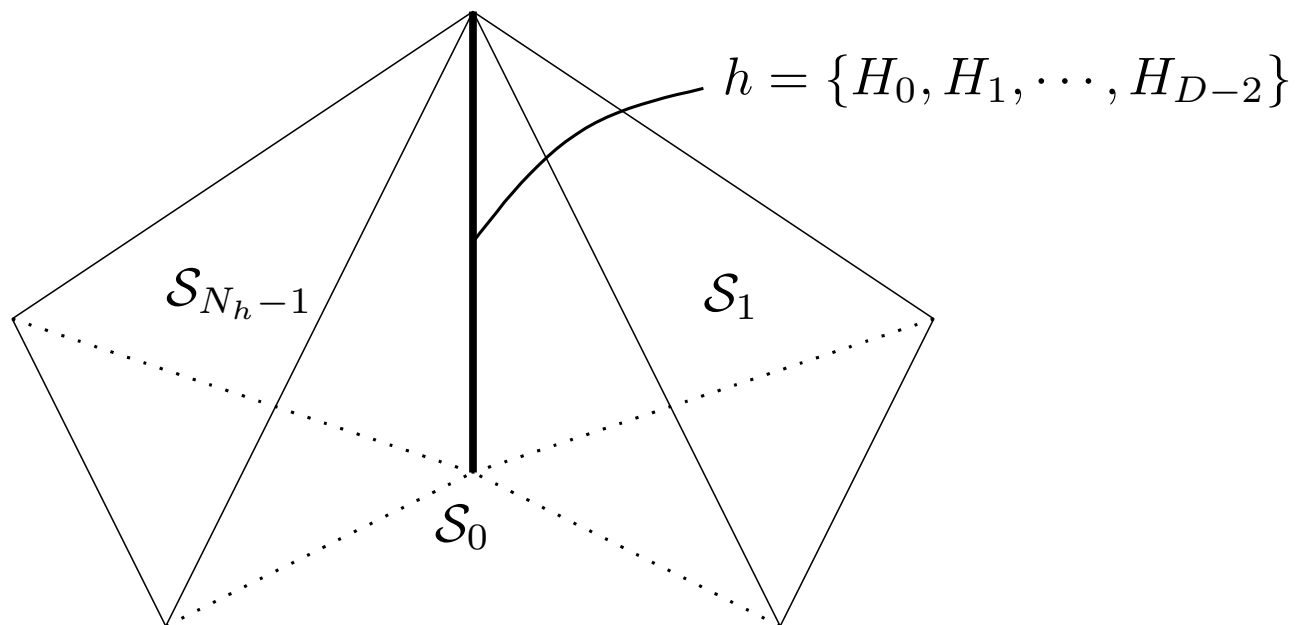
複体の構造

N_h 個の単体 $\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{N_h-1}\}$ から成る D 次元複体



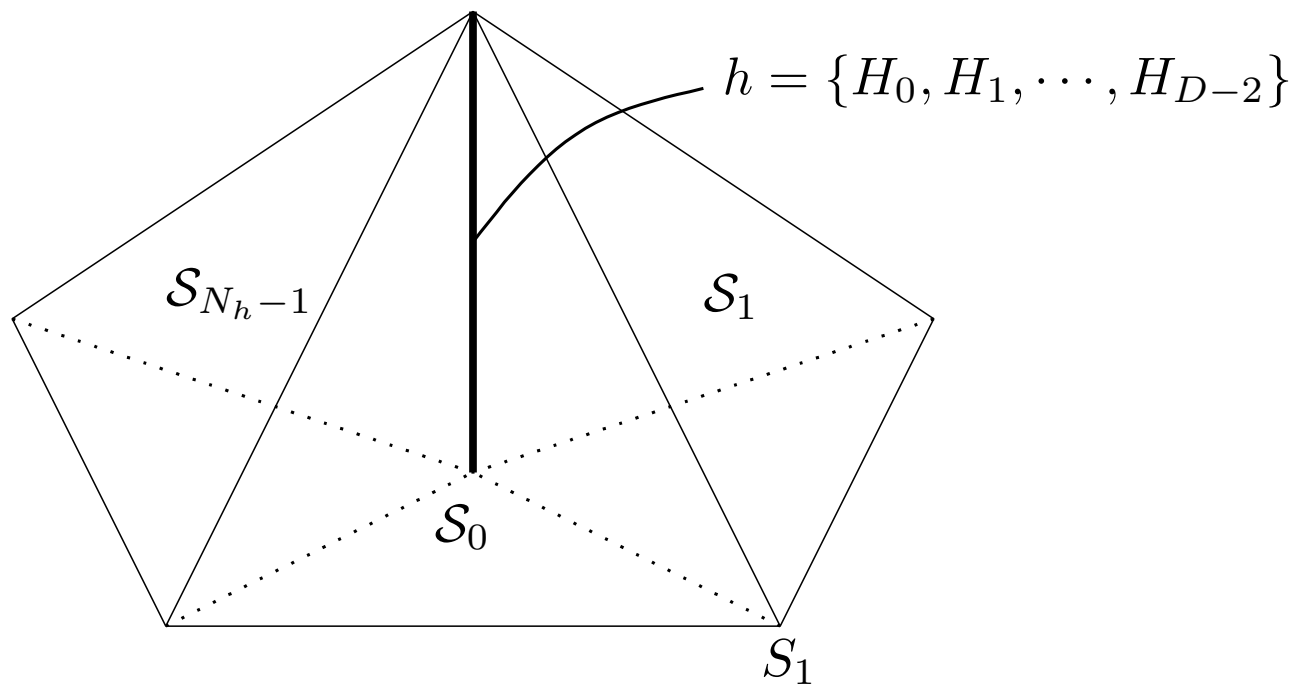
複体の構造

N_h 個の単体 $\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{N_h-1}\}$ から成る D 次元複体



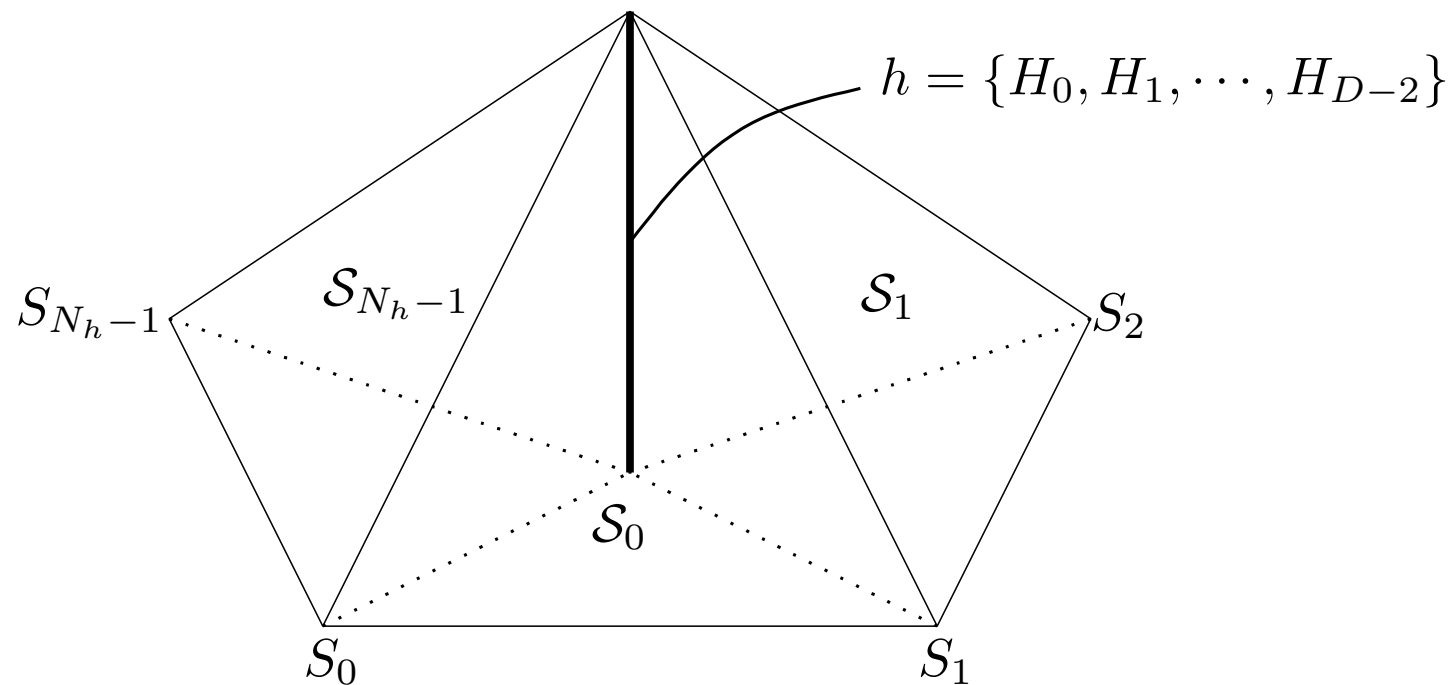
複体の構造

N_h 個の単体 $\{S_0, S_1, \dots, S_{N_h-1}\}$ から成る D 次元複体



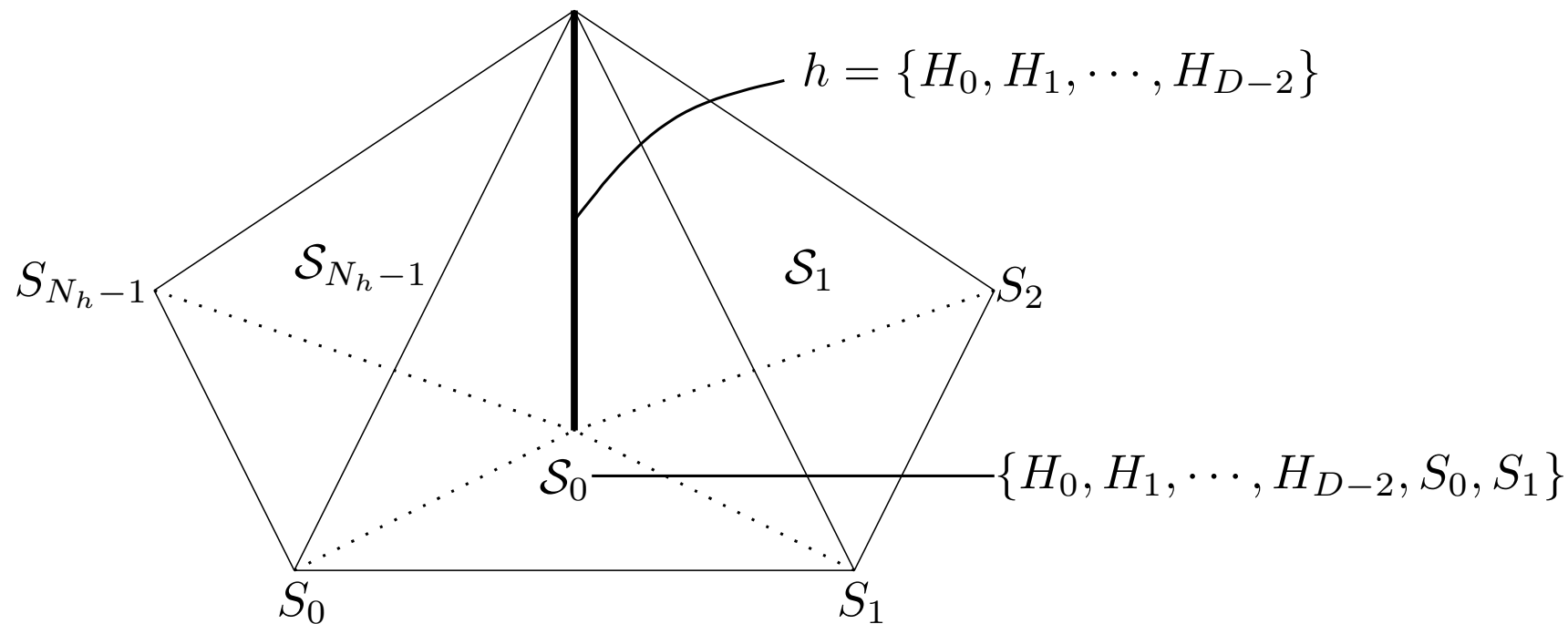
複体の構造

N_h 個の単体 $\{S_0, S_1, \dots, S_{N_h-1}\}$ から成る D 次元複体



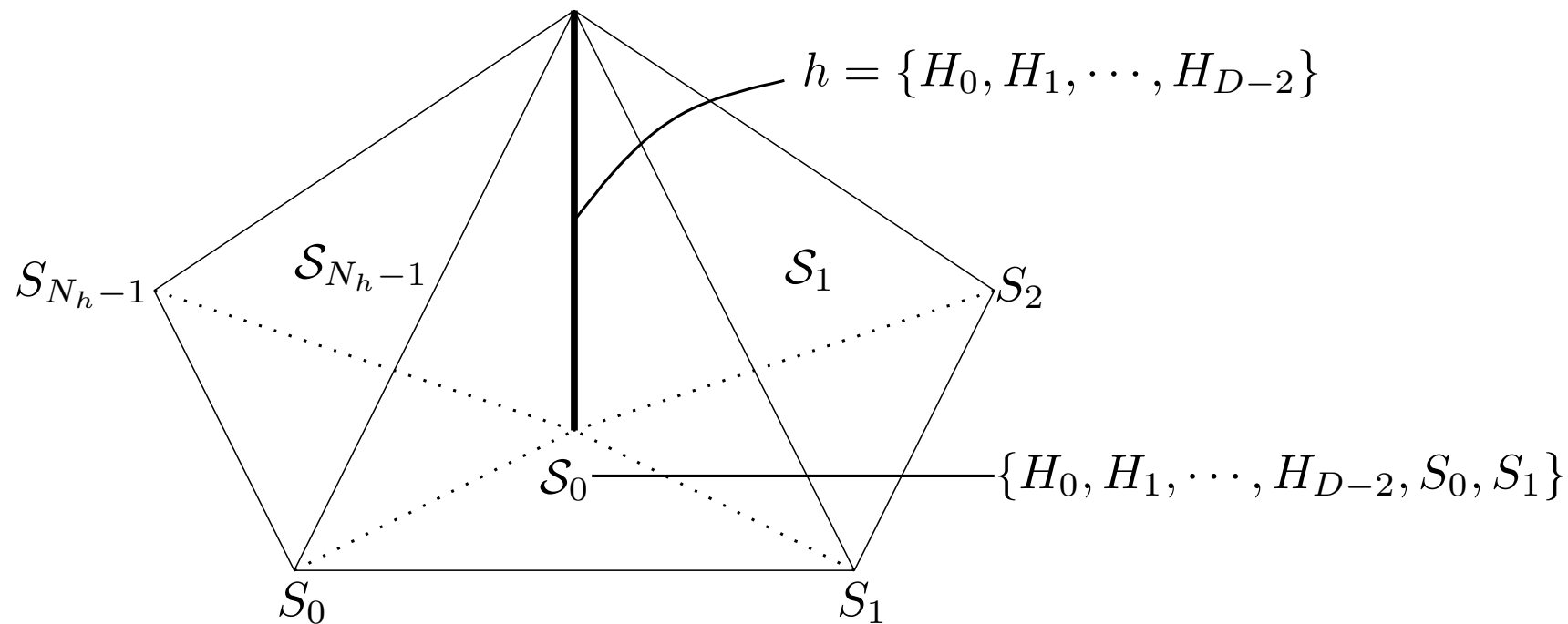
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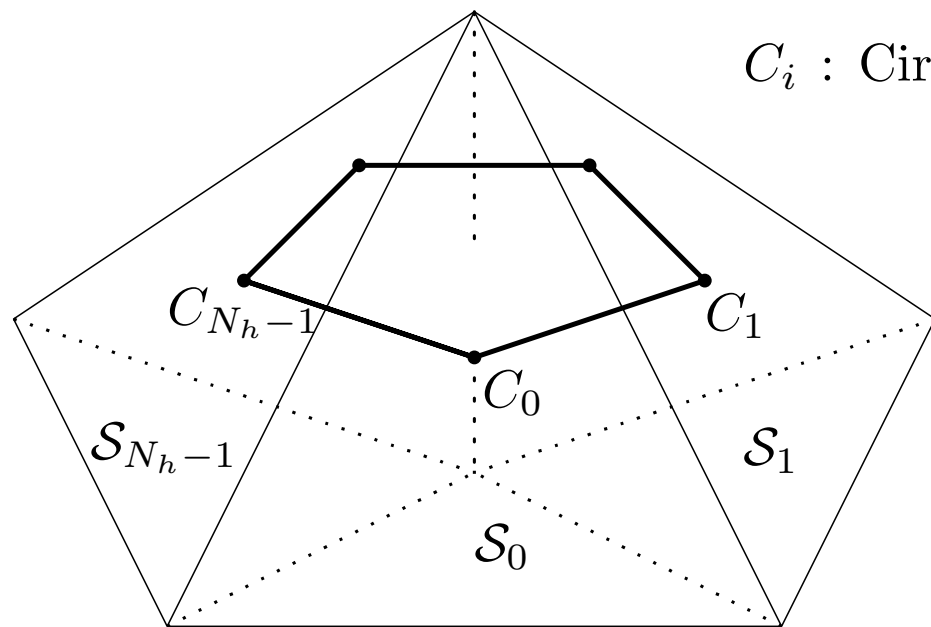
複体の構造

N_h 個の単体 $\{S_0, S_1, \dots, S_{N_h-1}\}$ から成る D 次元複体



$$S_i = \{H_0, H_1, \dots, H_{D-2}, S_i, S_{i+1 \bmod N_h}\}$$

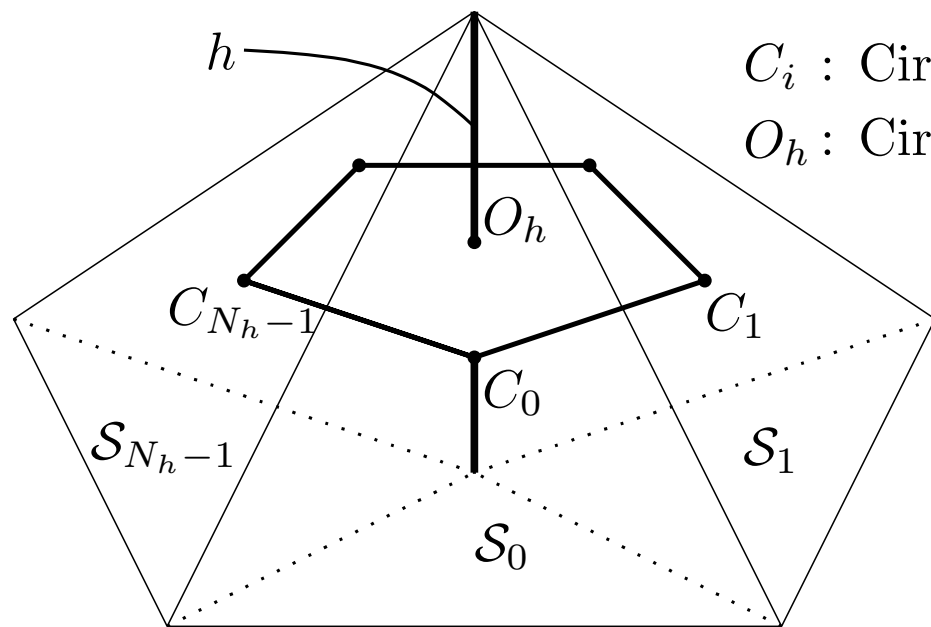
Voronoi ポリゴンの分解



C_i : Circumcenter of S_i

$$h^* = \{C_0, C_1, \dots, C_{N_h-1}\}$$

Voronoi ポリゴンの分解

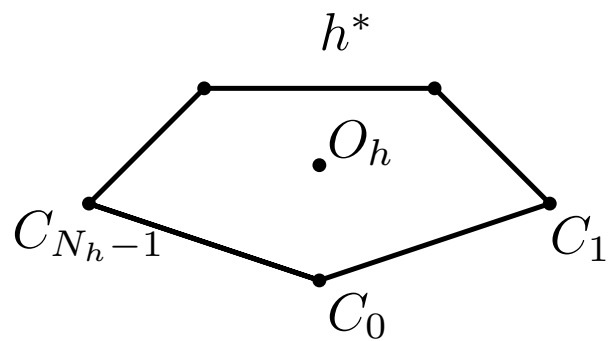


C_i : Circumcenter of \mathcal{S}_i

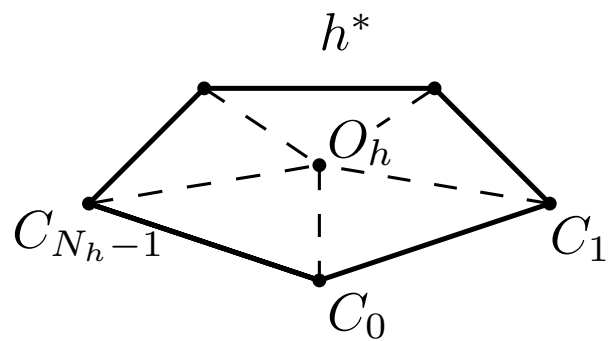
O_h : Circumcenter of Hinge

$$h^* = \{C_0, C_1, \dots, C_{N_h-1}\}$$

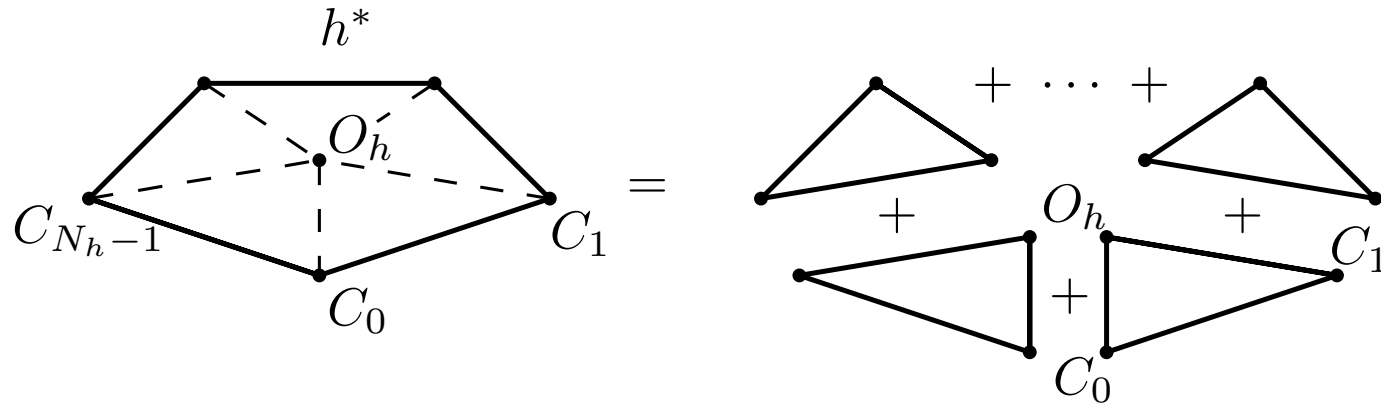
Voronoi ポリゴンの分解



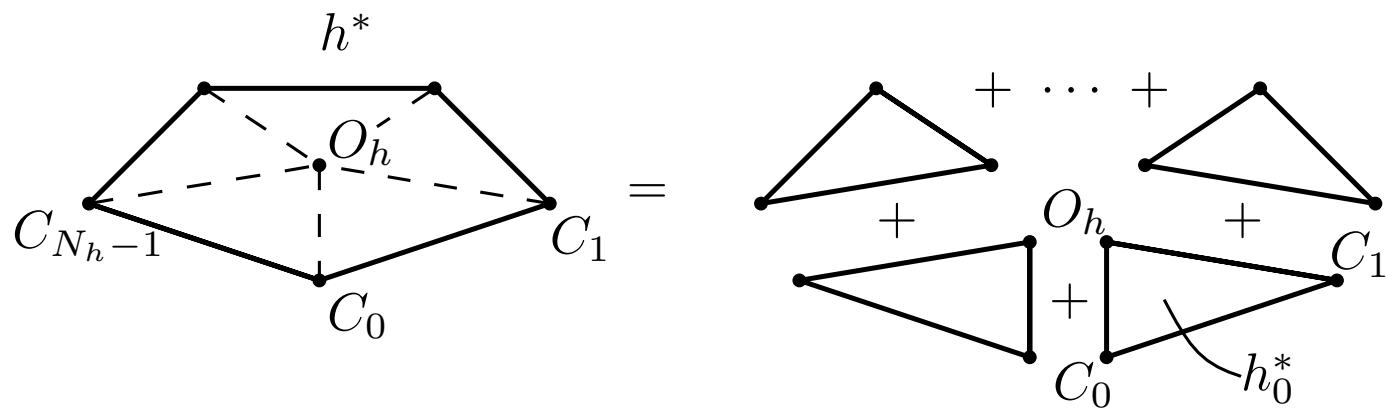
Voronoi ポリゴンの分解



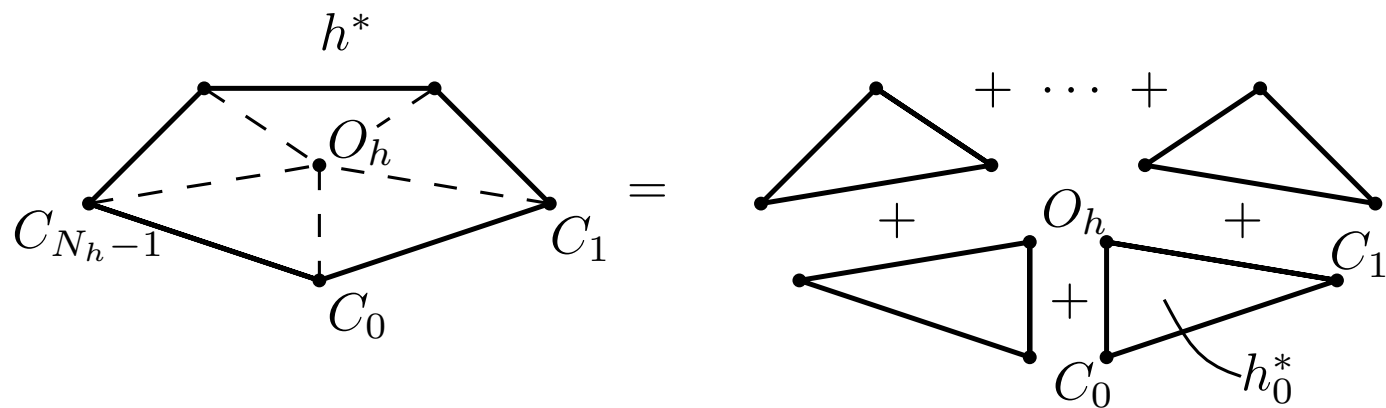
Voronoi ポリゴンの分解



Voronoi ポリゴンの分解



Voronoi ポリゴンの分解

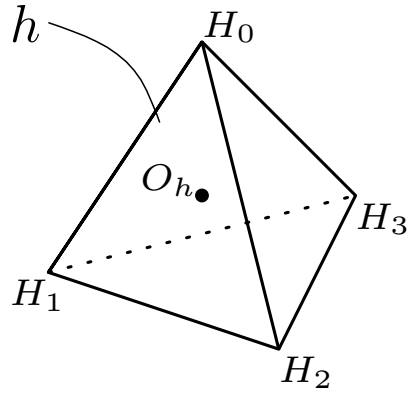


$$h_i^* = \{O_h, C_i, C_{i+1 \bmod N_h}\}$$

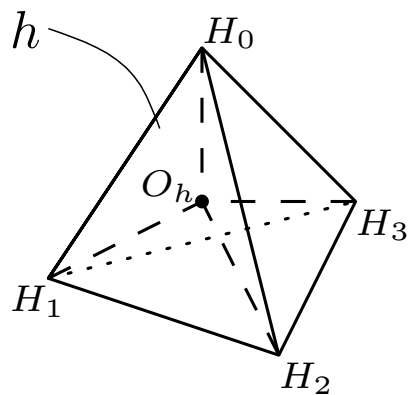
$A_{h_i}^*$: Area of h_i^*

$$A_h^* = \sum_{i=0}^{N_h-1} A_{h_i}^*$$

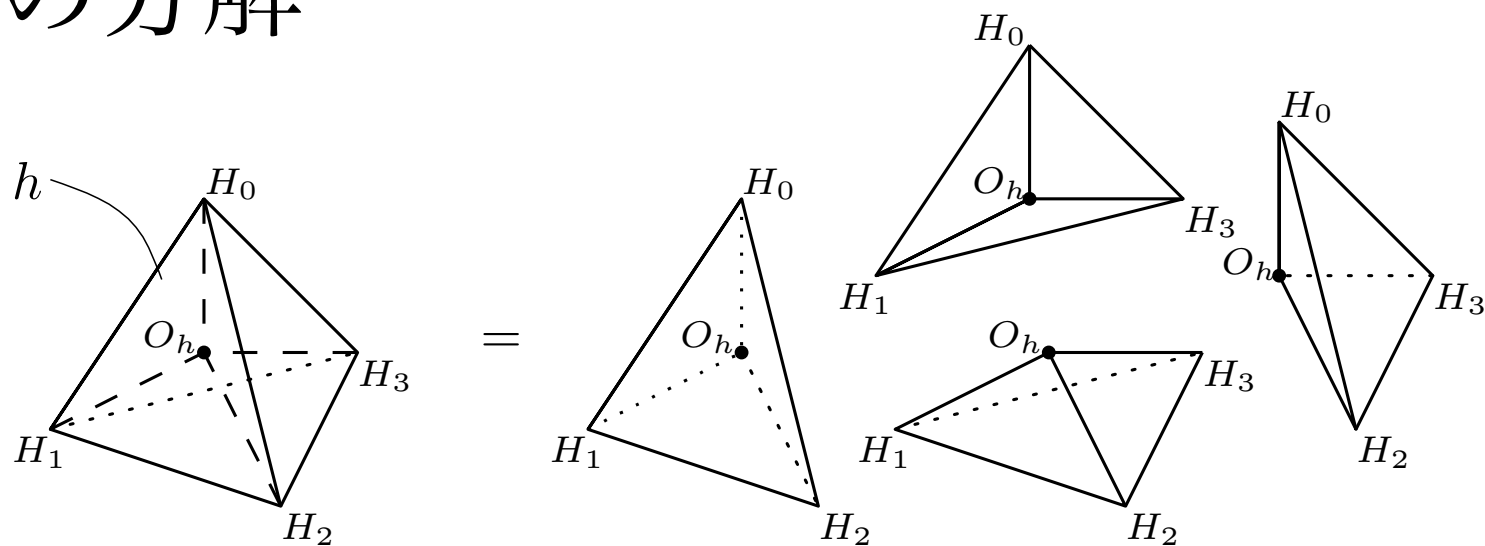
Hinge の分解



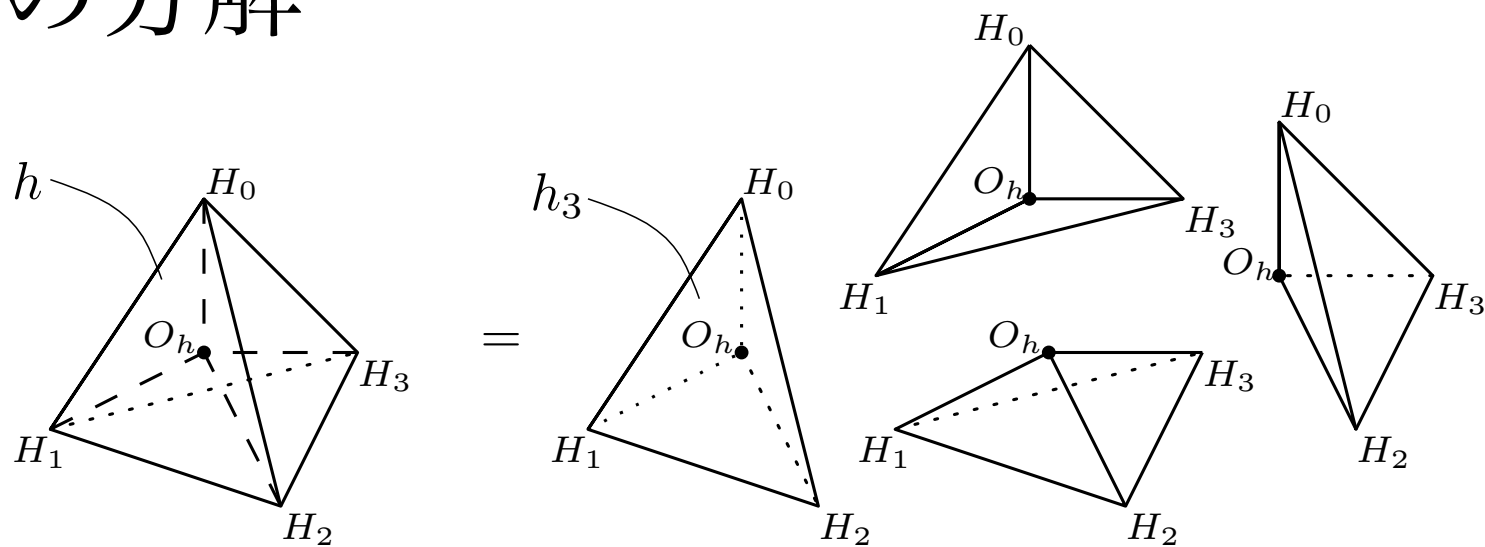
Hinge の分解



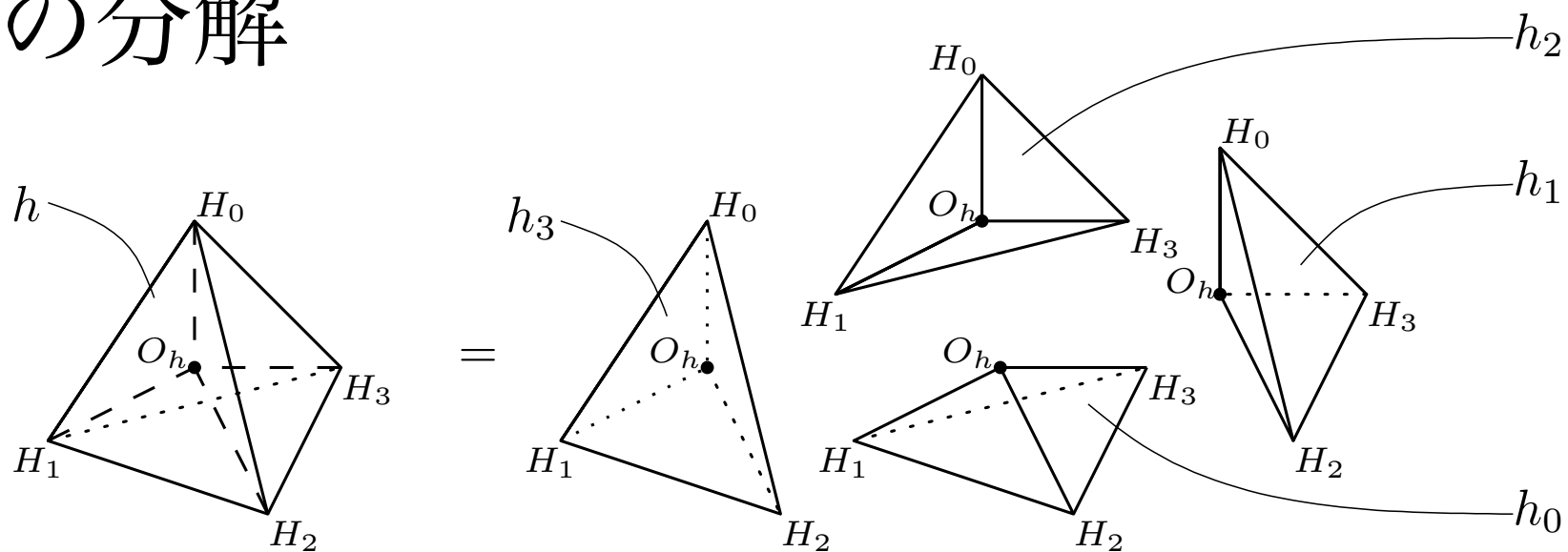
Hingeの分解



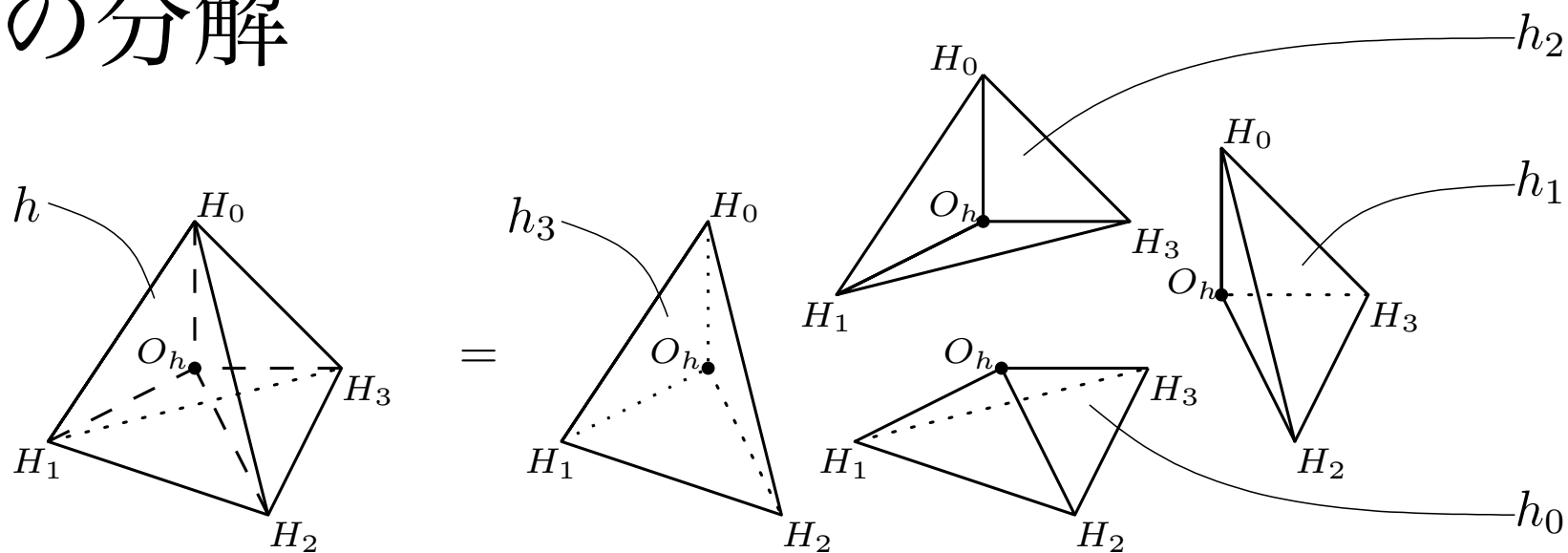
Hingeの分解



Hingeの分解



Hingeの分解

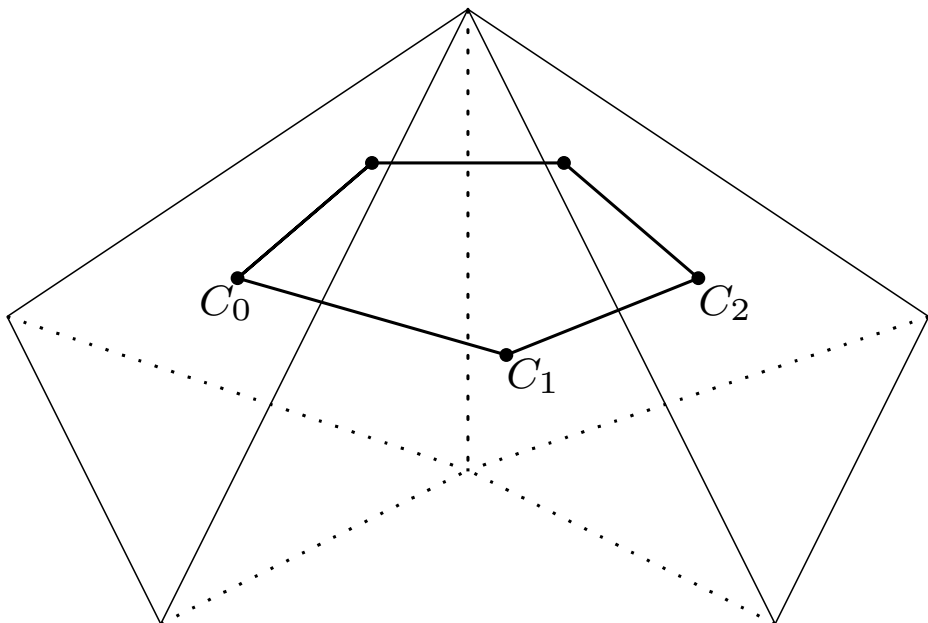


$$h_i = \{O_h, H_0, \dots, H_{i-1}, \wedge, H_{i+1}, \dots, H_{D-2}\}$$

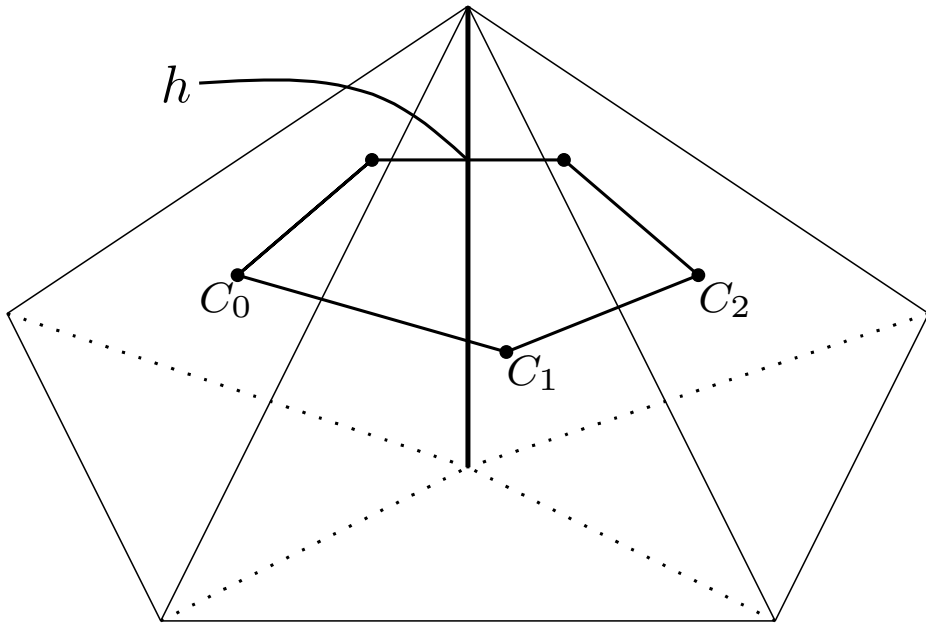
A_{h_i} : Volume of h_i

$$A_h = \sum_{i=0}^{D-2} A_{h_i}$$

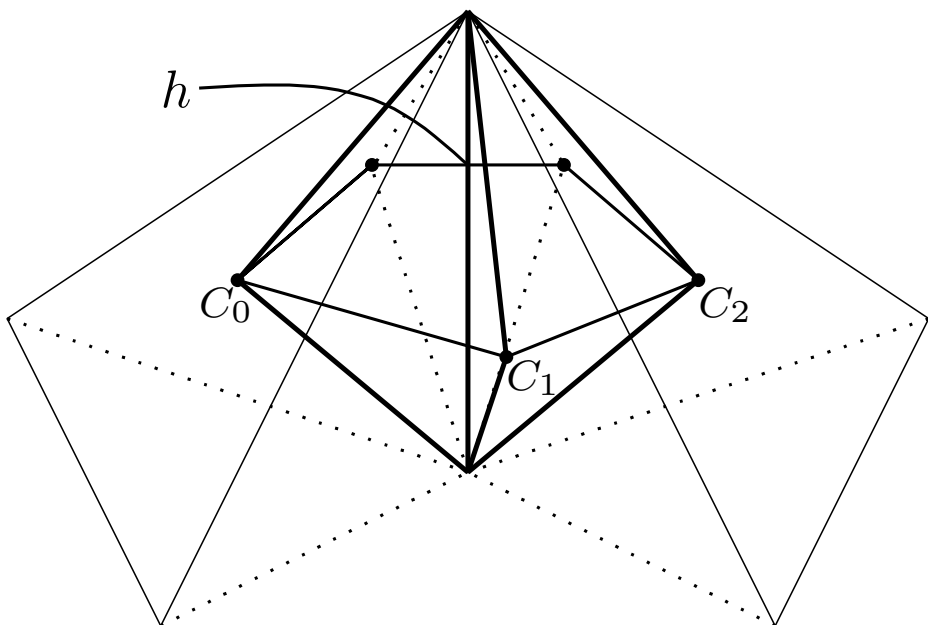
Proper Volume Element の定義



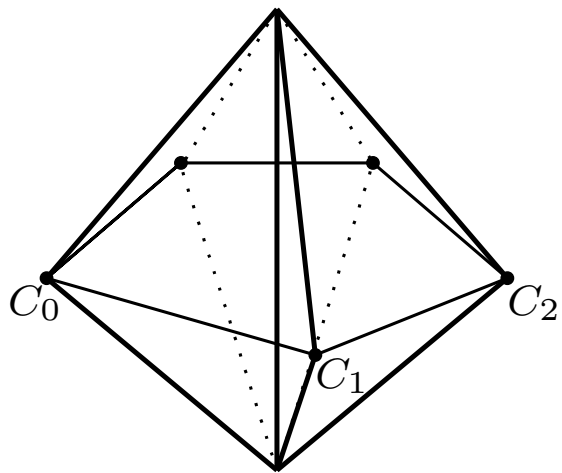
Proper Volume Element の定義



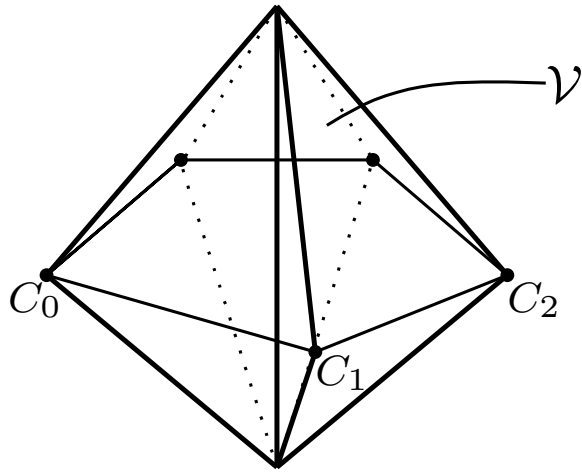
Proper Volume Element の定義



Proper Volume Element の定義

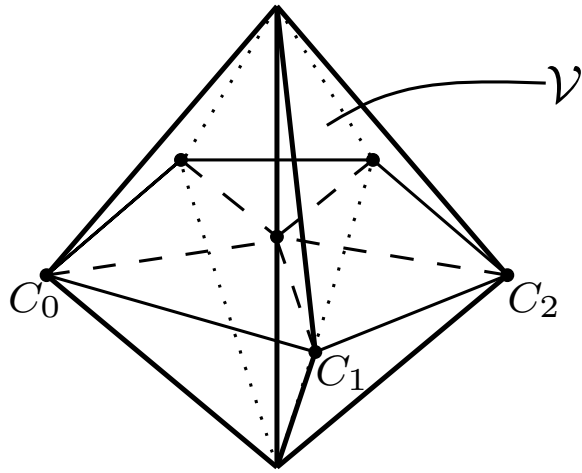


Proper Volume Element の定義



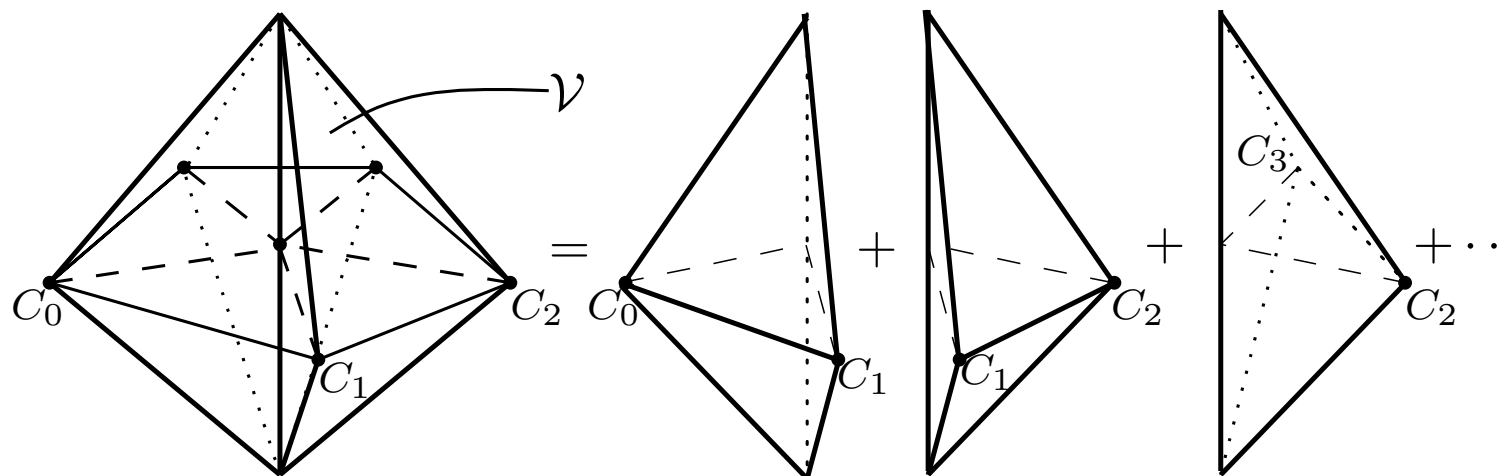
Volume: ΔV_{proper}

Proper Volume Element の定義



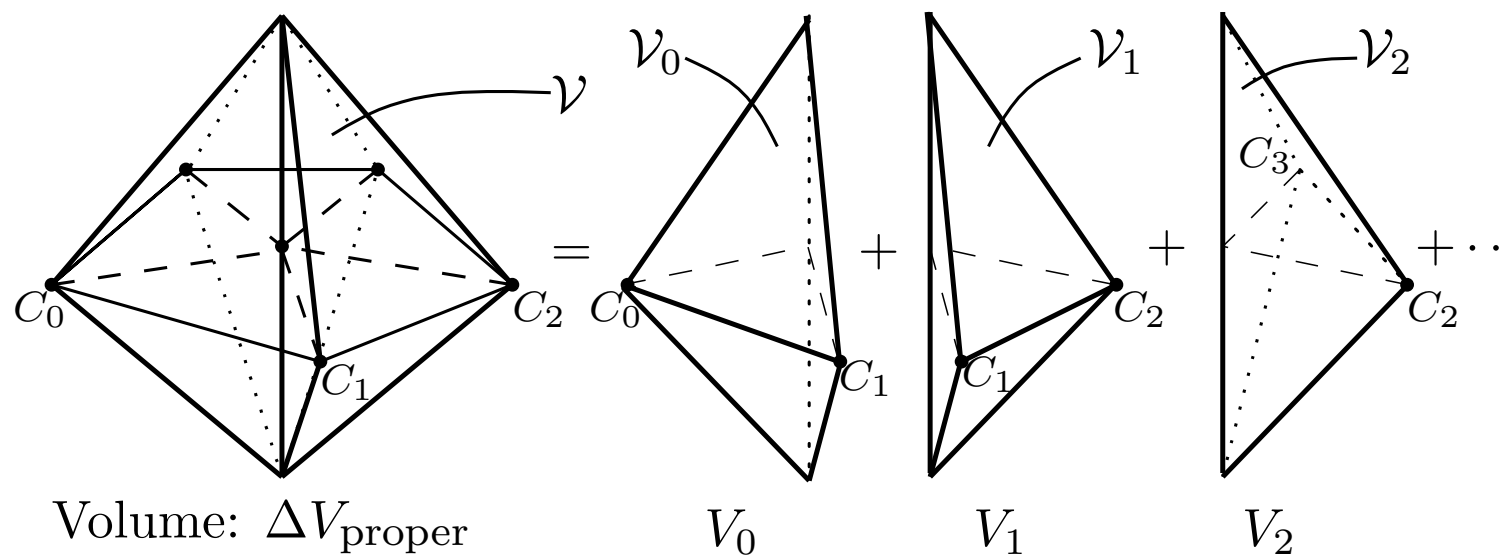
Volume: ΔV_{proper}

Proper Volume Element の定義

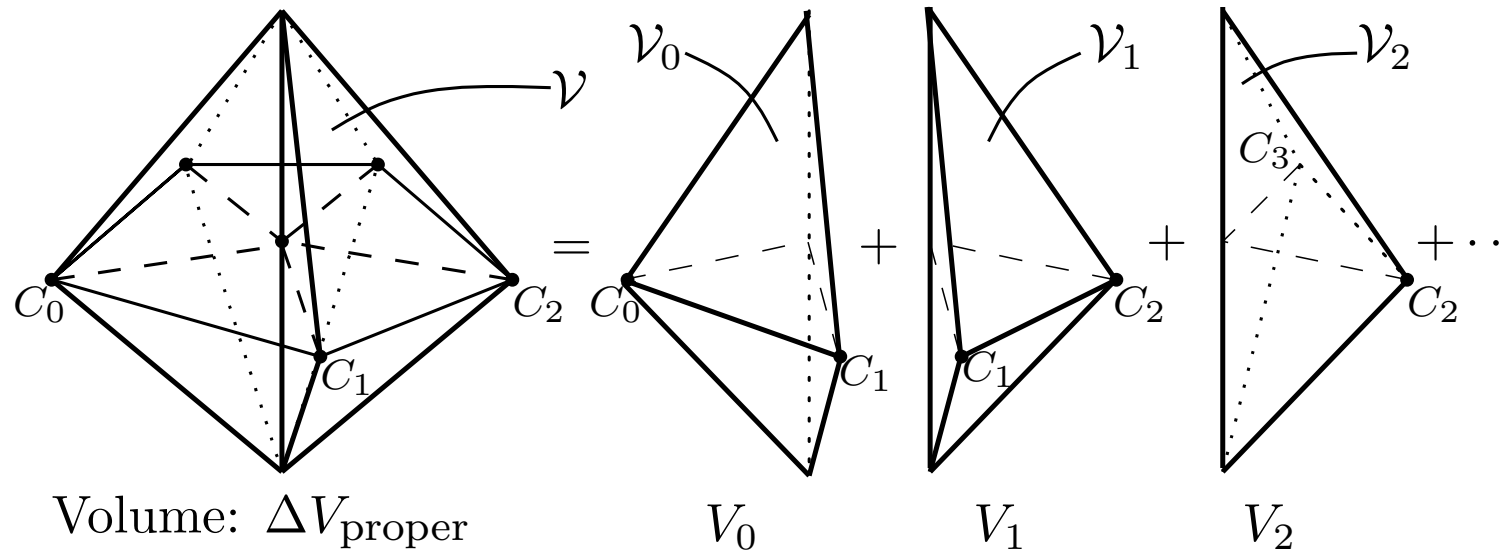


Volume: ΔV_{proper}

Proper Volume Element の定義



Proper Volume Element の定義



$$\mathcal{V}_i = \{H_0, H_1, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h}\}$$

$$\Delta V_{\text{proper}} = \sum_{i=0}^{N_h-1} V_i$$

\mathcal{V}_i の分解

$$\mathcal{V}_i = \{ H_0, H_1, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h} \}$$

\mathcal{V}_i の分解

$$\mathcal{V}_i = \{ \underline{H_0, H_1, \dots, H_{D-2}}, C_i, C_{i+1 \bmod N_h} \}$$

\mathcal{V}_i の分解

$$\begin{aligned}\mathcal{V}_i &= \{ \underline{H_0, H_1, \dots, H_{D-2}}, C_i, C_{i+1 \bmod N_h} \} \\ &= h \rightarrow \sum_{j=0}^{D-2} h_j = \sum_{j=0}^{D-2} \{ O_h, H_0, \dots, H_{j-1, \wedge}, H_{j+1}, \dots, H_{D-2} \}\end{aligned}$$

\mathcal{V}_i の分解

$$\begin{aligned}\mathcal{V}_i &= \{ \underline{H_0, H_1, \dots, H_{D-2}}, C_i, C_{i+1 \bmod N_h} \} \\ &= h \rightarrow \sum_{j=0}^{D-2} h_j = \sum_{j=0}^{D-2} \{ O_h, H_0, \dots, H_{j-1, \wedge}, H_{j+1}, \dots, H_{D-2} \}\end{aligned}$$

$$\mathcal{V}_{i_j} \equiv \{ O_h, H_0, \dots, H_{j-1, \wedge}, H_{j+1}, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h} \}$$

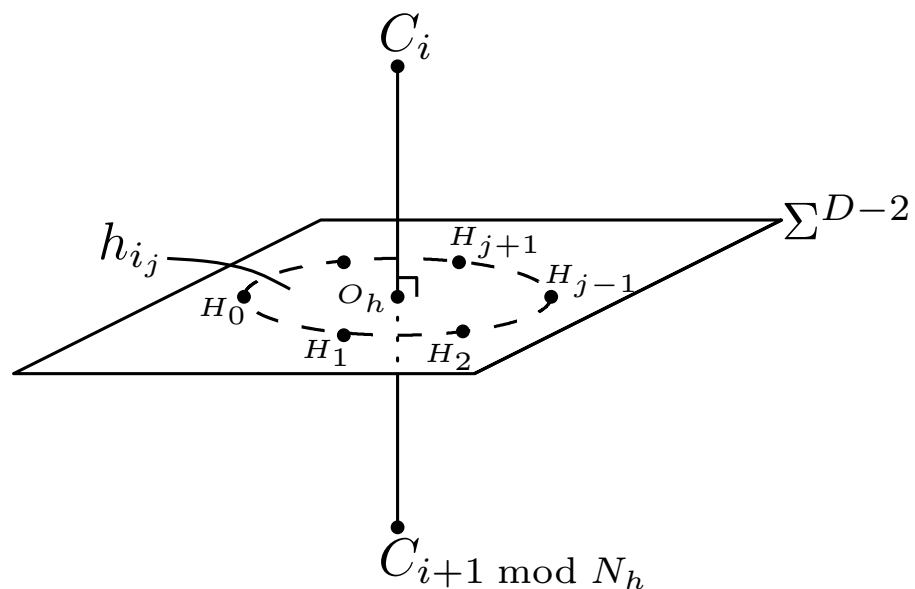
$$\text{Vol.} (\mathcal{V}_{i_j}) = V_{i_j}$$

$$V_i = \sum_{j=0}^{D-2} V_{i_j}$$

\mathcal{V}_{ij} の構造

$$\mathcal{V}_{ij} \equiv \{ \underline{O_h, H_0, \dots, H_{j-1}, \wedge, H_{j+1}, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h}} \}$$

頂点 $D + 1$ 個 $\dots \mathcal{V}_{ij}$ は D -単体



$$\overrightarrow{O_h C_i} \perp \overrightarrow{O_h H_j}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
 &\quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right)
 \end{aligned}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\underbrace{\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}}}_{\text{---}} \right) \\
 &\quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right)
 \end{aligned}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
 &\quad = (D-2)! A_{h_j} \\
 &\quad \quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right)
 \end{aligned}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
 &\quad = (D-2)! A_{h_j} \\
 &\quad \quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right)
 \end{aligned}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
 &\quad = (D-2)! A_{h_j} \\
 &\quad \quad \quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &\quad \quad \quad = 2A_{h_i}^*
 \end{aligned}$$

V_{ij} の値

$$\begin{aligned}
 V_{ij} &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
 &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &= \frac{1}{D!} \left(\overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
 &\quad = (D-2)! A_{h_j} \\
 &\quad \quad \quad \times \left(\overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
 &\quad \quad \quad = 2A_{h_i}^* \\
 &= \frac{2}{D(D-1)} A_{h_j} A_{h_i}^*
 \end{aligned}$$

Proper Volume Element in Regge Calculus

$$\begin{aligned}\Delta V_{\text{proper}} &= \sum_{i=0}^{N_h-1} \sum_{j=0}^{D-2} V_{ij} \\ &= \sum_{i=0}^{N_h-1} \sum_{j=0}^{D-2} \frac{2}{D(D-1)} A_{h_j} A_{h_i}^* \\ &= \frac{2}{D(D-1)} A_h A_h^*\end{aligned}$$

2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Grav. **14**, 199 (1997))

- Ricci Scalar Curvature
- Proper Volume Element
- **Hilbert Action**

Hilbert Action in Regge Calculus

$$\begin{aligned} S_{\text{Regge}} &= \frac{1}{16\pi} \sum_{\text{hinges}} \Delta V_{\text{proper}} R_{\text{Regge}} \\ &= \frac{1}{16\pi} \sum_{\text{hinges}} \left(\frac{2}{D(D-1)} A_h A_h^* \right) \left(D(D-1) \frac{\varepsilon_h}{A_h^*} \right) \\ &= \frac{1}{8\pi} \sum_{\text{hinges}} A_h \varepsilon_h \end{aligned}$$

3. Regge Calculus における Einstein 方程式

Regge Analogue of Metric

計量テンソル ... (大雑把に言うと) 時空の形を決めるもの

n -単体 ... $\frac{n(n+1)}{2}$ 本の辺の長さ全てを指定すると, 形が一意的に決定する.

※一般の立体ではこうはいかない

2次元の例: 三角形 ... 辺の長さを決めると形が決まる.

: 四角形 ... 辺の長さが等しい菱形が存在する.

基本変数: $g_{\mu\nu} \longrightarrow l_p$: Delaunay 格子に含まれる辺のうちの p 番目のものの長さ

Hilbert Action の変分

(L. Schläfli, Quart. J. Pure Appl. Math. **2**, 269 (1858))

$$\frac{\partial S_{\text{Regge}}}{\partial l_p} = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial}{\partial l_p} (A_h \varepsilon_h) = \frac{1}{8\pi} \left[\sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h + \sum_{\text{hinges}} A_h \frac{\partial \varepsilon_h}{\partial l_p} \right]$$

$= 0$ (Schläfli Identity)

真空の Einstein 方程式

$$0 = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h$$

※ Schläfli Identity は 表面項が Gauss の定理によって落とされるのに似ている。しかし両者の関係はあまり分かっていない。

4. Expanding Polyhedral Universe

... Applying the Regge Calculus to $(2 + 1)$ -dimensional closed FLRW universe

(P. A. Collins and R. M. Williams, Phys. Rev. **D7**, 965 (1973))

(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 024032 (2016))

(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 023502 (2016))

準備: 任意格子への拡張

Delaunay 格子 (単体格子) について Hilbert Action を組んだが, 実は任意の格子に対して同様の形式が適用可能であることが分かっている.

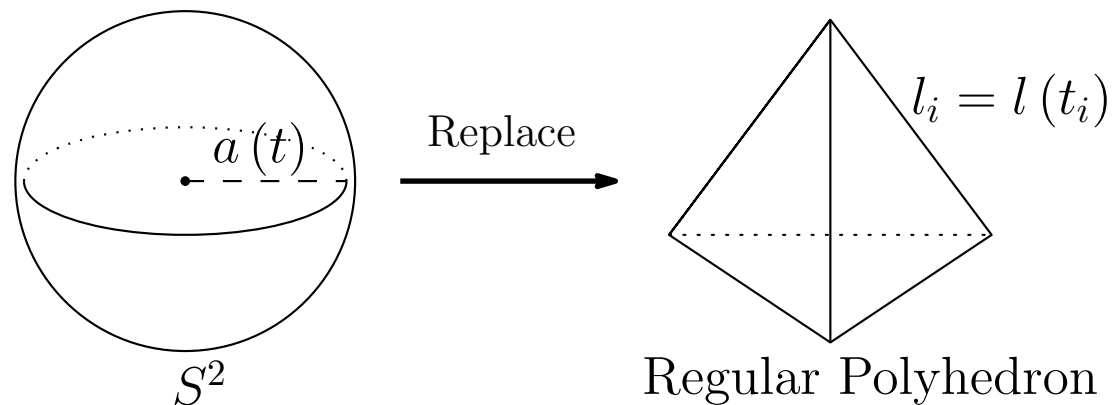
さらに宇宙項を入れると

$$S_{\text{Regge}} = \frac{1}{8\pi} \left(\sum_{i \in \{\text{hinges}\}} A_i \varepsilon_i - \Lambda \sum_{i \in \{\text{blocks}\}} V_i \right)$$

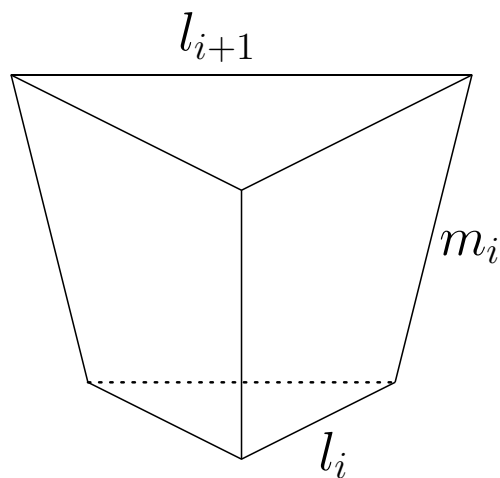
- V_i : 任意の形の格子胞 (Block) の体積

Regular Polyhedral Universe

球面状の Cauchy surface を正多面体で置き換える。

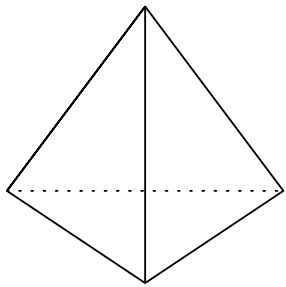


Block は正多角形を底面にもつ錐台



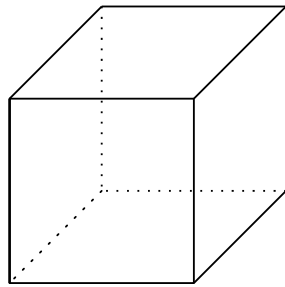
Schläfli Symbols

There are only five types of regular polyhedrons specified by the Schläfli symbols



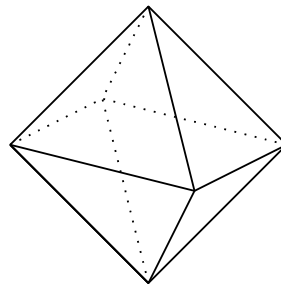
$\{3, 3\}$

Tetrahedron



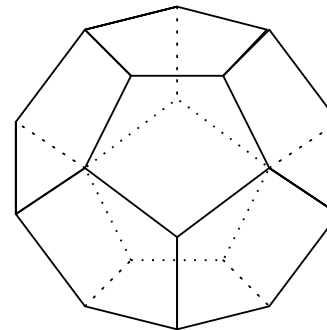
$\{4, 3\}$

Cube



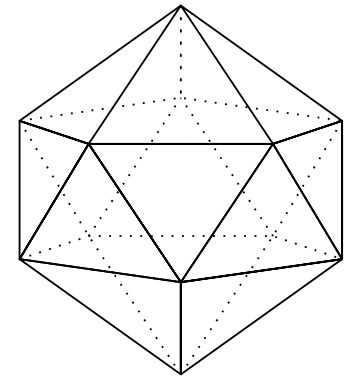
$\{3, 4\}$

Octahedron



$\{5, 3\}$

Dodecahedron



$\{3, 5\}$

Icosahedron

Schläfli Symbols $\{p, q\}$... p : the number of the sides of a face
 q : the number of faces around each vertex

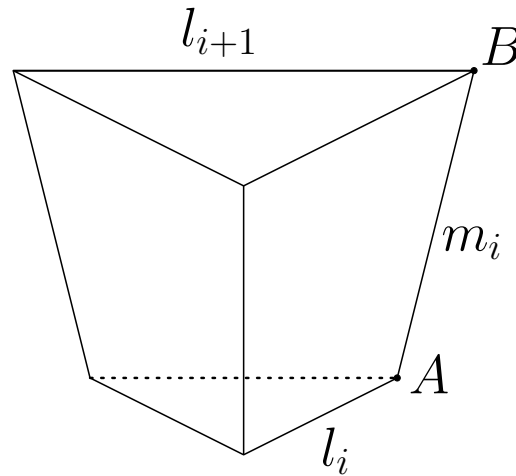
Equations of motion

$$\begin{aligned}
 2\pi - q \arccos \left(-\frac{4m_i^2 \cos \frac{2\pi}{p} + \delta l_i^2}{4m_i^2 - \delta l_i^2} \right) &= \frac{q\Lambda}{12} \frac{(l_{i+1}^2 + l_{i+1}l_i + l_i^2)m_i}{\sqrt{m_i^2 - \frac{1}{4}\delta l_i^2 \csc^2 \frac{\pi}{p}}} \cot \frac{\pi}{p}, \\
 \arccos \frac{\delta l_i \cot \frac{\pi}{p}}{\sqrt{4m_i^2 - \delta l_i^2}} - \arccos \frac{\delta l_{i-1} \cot \frac{\pi}{p}}{\sqrt{4m_{i-1}^2 - \delta l_{i-1}^2}} &= \frac{\Lambda}{12} \left[\frac{(l_{i+1} + 2l_i)m_i^2 + \frac{3}{4}l_i^2 \delta l_i \csc^2 \frac{\pi}{p}}{\sqrt{m_i^2 - \frac{1}{4}\delta l_i^2 \csc^2 \frac{\pi}{p}}} \right. \\
 &\quad \left. + \frac{(2l_i + l_{i-1})m_{i-1}^2 - \frac{3}{4}l_i^2 \delta l_{i-1} \csc^2 \frac{\pi}{p}}{\sqrt{m_{i-1}^2 - \frac{1}{4}\delta l_{i-1}^2 \csc^2 \frac{\pi}{p}}} \right] \cot \frac{\pi}{p}.
 \end{aligned}$$

To compare with the continuum theory, we introduce a circumradius of the polyhedron as an analogue of the scale factor

$$a_{Ri} = \frac{l_i \sin \frac{\pi}{q}}{2\sqrt{\sin^2 \frac{\pi}{p} - \cos^2 \frac{\pi}{q}}}$$

Strut length m_i はなにか?



点 A と点 B は空間的には同一

→ m_i は多面体の頂点に置かれた時計が示す経過固有時と考え

$$m_i = i\delta t_i$$

とする.

Continuum time limit

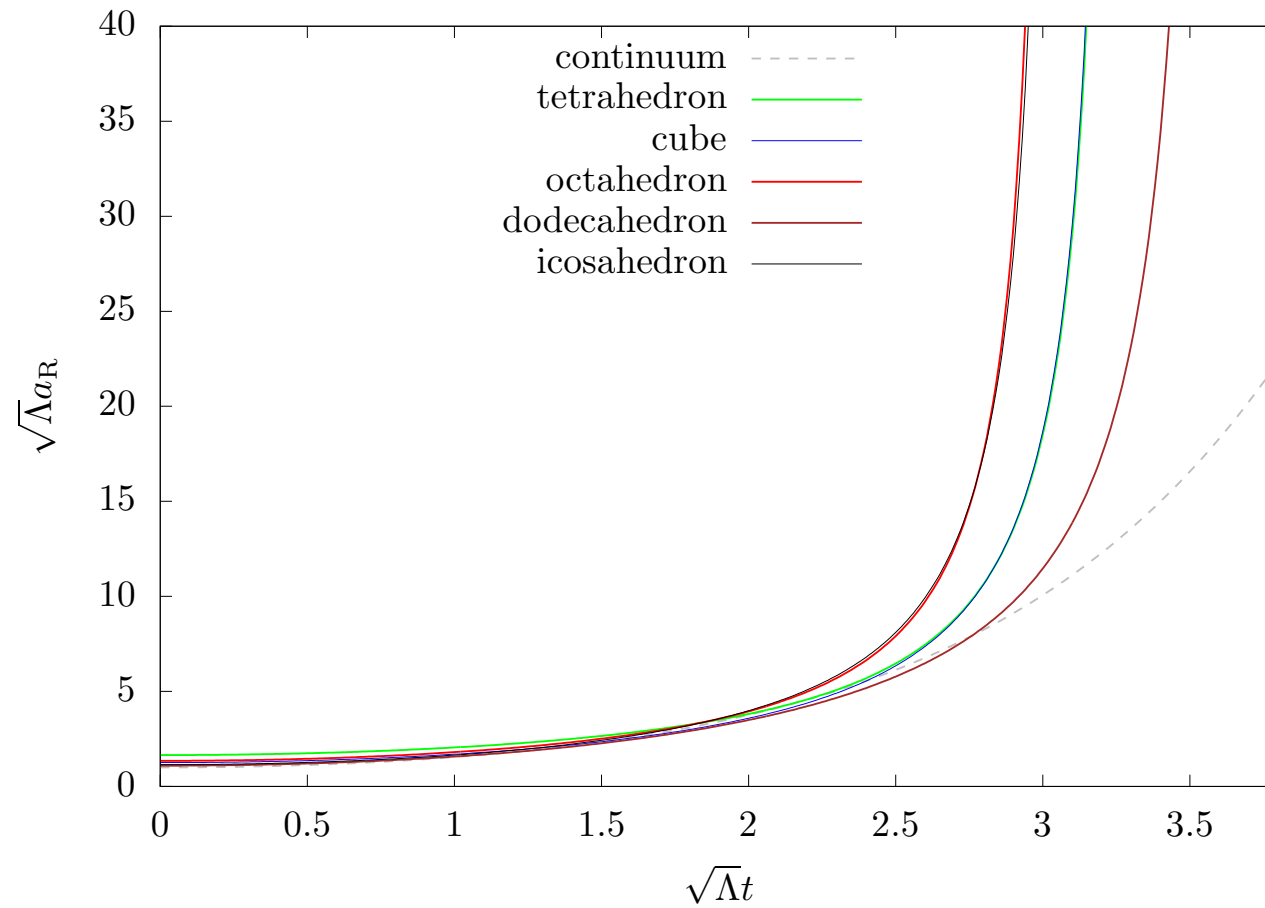
$$m_i = i\delta t_i \rightarrow idt$$

$$2\pi - q \arccos \left(\frac{\dot{l}^2 - 4 \cos \frac{2\pi}{p}}{4 + \dot{l}^2} \right) = \frac{q\Lambda}{2} \frac{l^2 \cos \frac{\pi}{p}}{\sqrt{4 \sin^2 \frac{\pi}{p} + \dot{l}^2}},$$

$$\frac{\ddot{l}}{4 + \dot{l}^2} = \frac{\Lambda}{4} l \left[1 - \frac{l\ddot{l}}{2 \left(4 \sin^2 \frac{\pi}{p} + \dot{l}^2 \right)} \right],$$

$$a_R(t) = \frac{l(t) \sin \frac{\pi}{q}}{2 \sqrt{\sin^2 \frac{\pi}{p} \cos^2 \frac{\pi}{q}}}$$

Numerical Solution

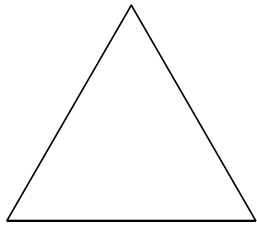


- finite-precision approximation of FLRW
- expands to infinite within finite time

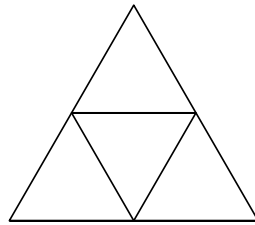
Geodesic Domes

Frequency ν ... Degree of subdivision

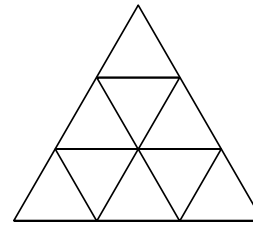
frequency 1



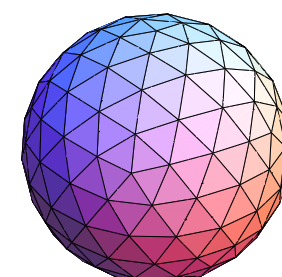
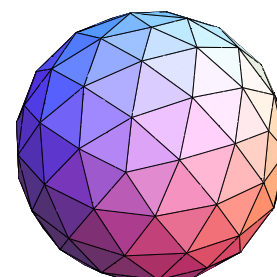
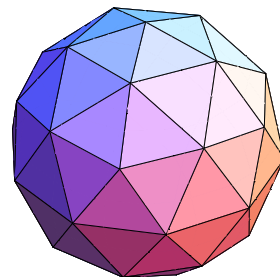
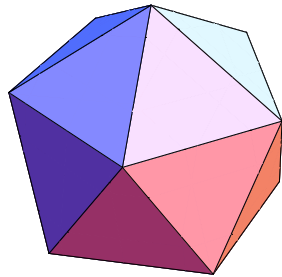
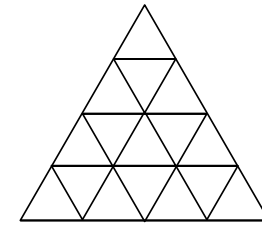
frequency 2



frequency 3



frequency 4



Better approximates a sphere than the regular polyhedrons.

Equations of Motion for Geodesic Domes

$$\nu = 2$$

$$\frac{1}{5}\varepsilon_1 + \frac{1}{2}\varepsilon_2 = \Lambda l^2 \left[\frac{\sin \xi \cos \frac{\xi}{2}}{\sqrt{4 \cos^2 \frac{\xi}{2} + l^2}} + \frac{\sin^2 \frac{\xi}{2}}{\sqrt{3 + 4l^2 \sin^2 \frac{\xi}{2}}} \right]$$

$$\xi = \text{const.}$$

$$\varepsilon_1 = 2\pi - 5\theta_{1,1}$$

$$\varepsilon_2 = 2\pi - 4\theta_{2,1} - 2\theta_{2,2}$$

$$\theta_{1,1} = \arccos \frac{4 \cos \xi + l^2}{4 + l^2}$$

$$\theta_{2,1} = \arccos \frac{(2 + l^2) \sin \frac{\xi}{2}}{\sqrt{(4 + l^2) (1 + l^2 \sin^2 \frac{\xi}{2})}}$$

$$\theta_{2,2} = \arccos \frac{1 + 2l^2 \sin^2 \frac{\xi}{2}}{2 (1 + l^2 \sin^2 \frac{\xi}{2})}$$

Equations of Motion for Geodesic Domes

$$\nu = 3$$

$$\frac{1}{5}\varepsilon_1 + \varepsilon_2 + \frac{1}{3}\varepsilon_3 = \Lambda l^2 \left[\frac{\sin \xi \cos \frac{\xi}{2}}{\sqrt{4 \cos^2 \frac{\xi}{2} + i^2}} + \frac{2 \sin^2 \frac{\xi}{2} \cos^2 \frac{\eta}{2}}{\sqrt{\sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}} + \frac{\sin^2 \frac{\xi}{2} \csc \frac{\eta}{2} \sin \zeta \cos \zeta}{\sqrt{4 \sin^2 \frac{\eta}{2} \cos^2 \frac{\zeta}{2} + i^2 \sin^2 \frac{\xi}{2}}} \right]$$

$$\xi = \text{const.}$$

$$\eta = \text{const.}$$

$$\zeta = \text{const.}$$

$$\varepsilon_1 = 2\pi - 5\theta_{1,1}$$

$$\varepsilon_2 = 2\pi - 4\theta_{2,1} - 2(\theta_{2,1} + \theta_{2,2} + \theta_{2,3})$$

$$\varepsilon_3 = 3(\theta_{3,2} + \theta_{3,3})$$

$$\theta_{1,1} = \arccos \frac{4 \cos \xi + i^2}{4 + i^2}, \quad \theta_{2,1} = \arccos \frac{(2 + i^2) \sin \frac{\xi}{2}}{\sqrt{(4 + i^2) (1 + i^2 \sin^2 \frac{\xi}{2})}}$$

$$\theta_{2,2} = \arccos \frac{2 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{\sqrt{(1 + i^2 \sin^2 \frac{\xi}{2}) (4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2})}}, \quad \theta_{2,3} = \arccos \frac{(2 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}) \sin \frac{\xi}{2}}{\sqrt{(4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}) (\sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2} \sin^2 \frac{\zeta}{2})}}$$

$$\theta_{3,2} = \arccos \frac{4 \cos \eta \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}, \quad \theta_{3,3} = \arccos \frac{4 \cos \zeta \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}$$

Shortcomings of Geodesic Domes

- Frequency を上げると球面への近似精度は向上するが、考慮すべきパラメータの数が指数関数的に増加する。
- $\nu \rightarrow \infty$ の極限では無限個のパラメータが必要になる。
⇒ 解析的に連続極限を求めることが不可能。

Pseudo-Regular Polyhedron Universe

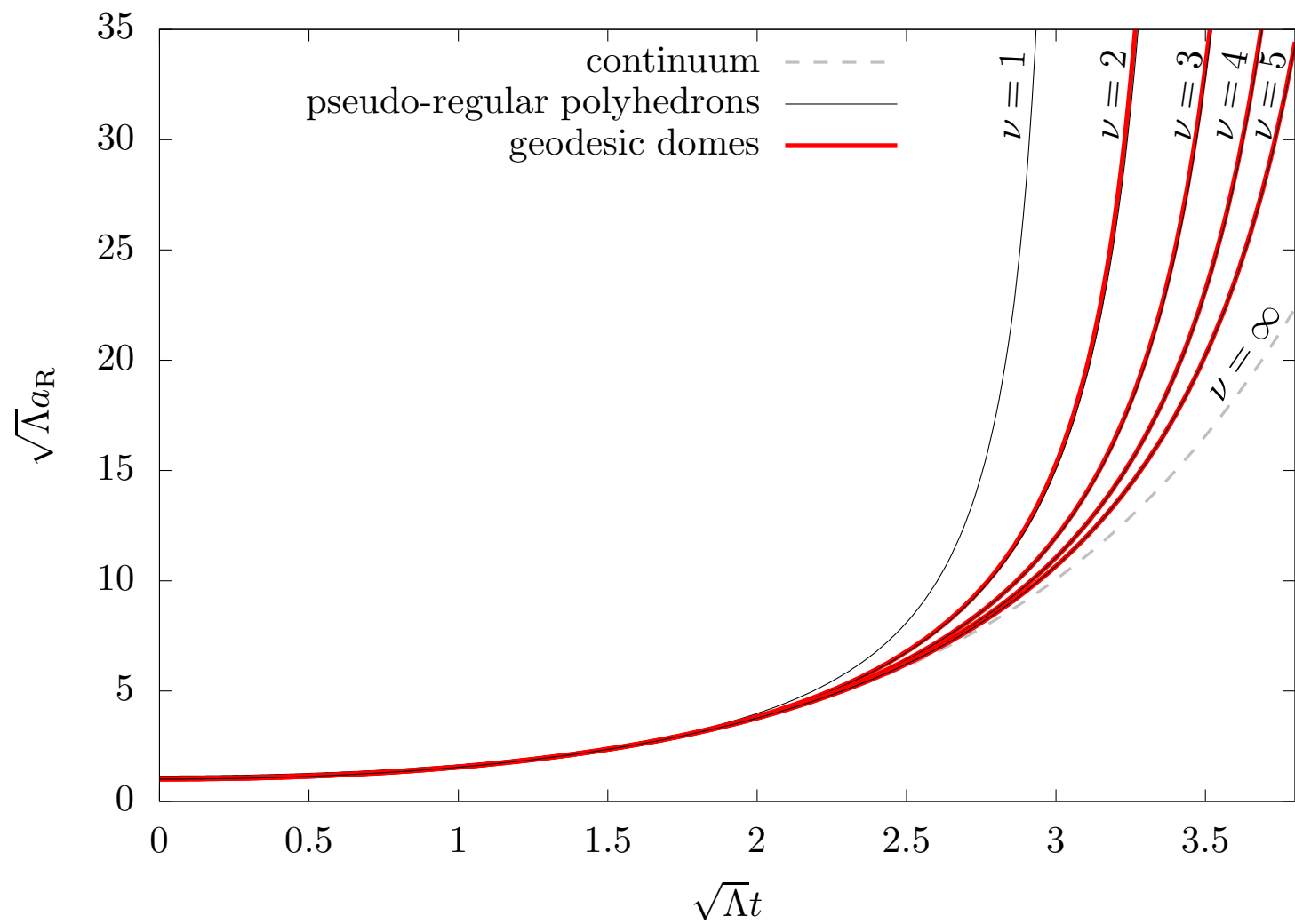
Regarding the geodesic domes as pseudo-regular polyhedrons, we may define the fractional Schläfli Symbol

$$\{p, q\} = \left\{ 3, \frac{30\nu^2}{5\nu^2 + 1} \right\}$$

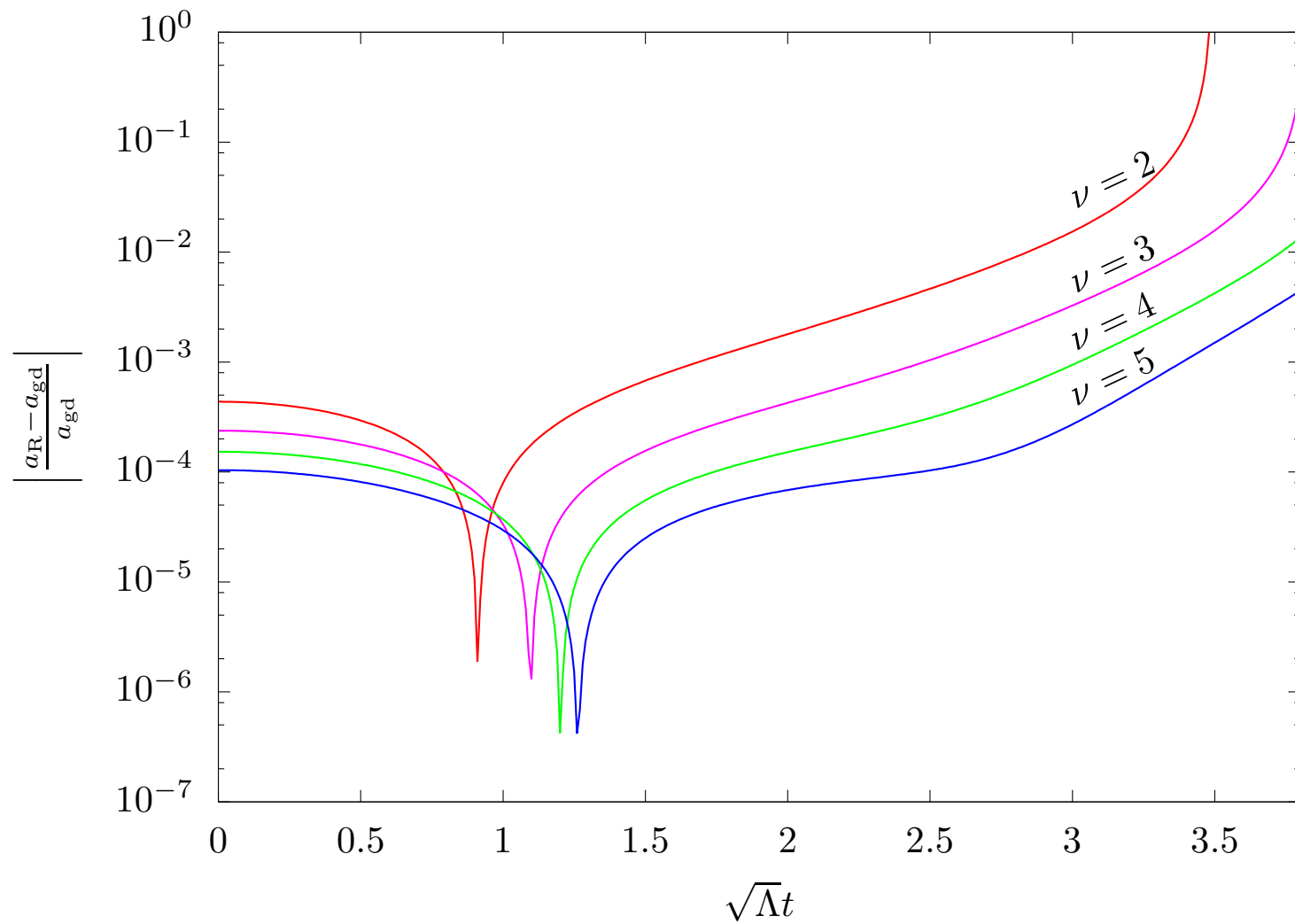
Equation of motion

$$\frac{\ddot{l}}{4 + \dot{l}^2} = \frac{\Lambda}{4} l \left[1 - \frac{l\ddot{l}}{2(3 + \dot{l}^2)} \right],$$
$$a_R(t) = \frac{l(t)}{\sqrt{3}} \tan \frac{(5\nu^2 + 1)\pi}{30\nu^2}$$

Plots of Scale Factors



Deviation $\left| \frac{a_R - a_{gd}}{a_{gd}} \right|$



Infinite Frequency Limit

Taking the limit $\nu \rightarrow \infty$

$$\left\{ \begin{array}{l} \frac{\ddot{l}}{4+l^2} = \frac{\Lambda l}{4} \left[1 - \frac{\ddot{l}}{2(3+l^2)} \right] \\ a_R(t) = \frac{l(t)}{\sqrt{3}} \tan \frac{(5\nu^2+1)\pi}{30\nu^2} \end{array} \right.$$

$$\implies \ddot{a}_R = \Lambda a_R$$

Exactly coincides with Einstein equations in General relativity.

5. Summary

Summary

Regge Calculus

- 単体分割による計算の手法の一つ.
- 各単体の形状を変化させることで多様体の Dynamics を表現.

Hilbert 作用

$$S_{\text{Regge}} = \frac{1}{16\pi} \sum_{\text{hinges}} \Delta V_{\text{proper}} R_{\text{Regge}} = \frac{1}{8\pi} \sum_{\text{hinges}} A_h \varepsilon_h$$

A_h : Hinge Volume

ε_h : Deficit Angle

真空の Einstein 方程式

$$0 = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h$$

Summary

Polyhedral Universe

- Geodesic Dome Model
 - 任意の精度で球面を近似.
 - Frequency を上げるごとに方程式が複雑化.
 - 連続極限への解析的アプローチ不可能.
- Pseudo-Regular Polyhedron Model
 - Geodesic Dome を非常に高精度で近似.
 - Frequency によらず方程式が単一かつ簡潔.
 - 連続極限を解析的に求めることが可能.