

# Oscillating 4-Polytopal Universe

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## in Regge Calculus

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2. Regge Calculus における Hilbert 作用
3. Regge Calculus における Einstein 方程式
4. Expanding Polyhedral Universe
5. Oscillating 4-Polytopal Universe
6. Summary and Future work

# 1. Introduction: 单体分割

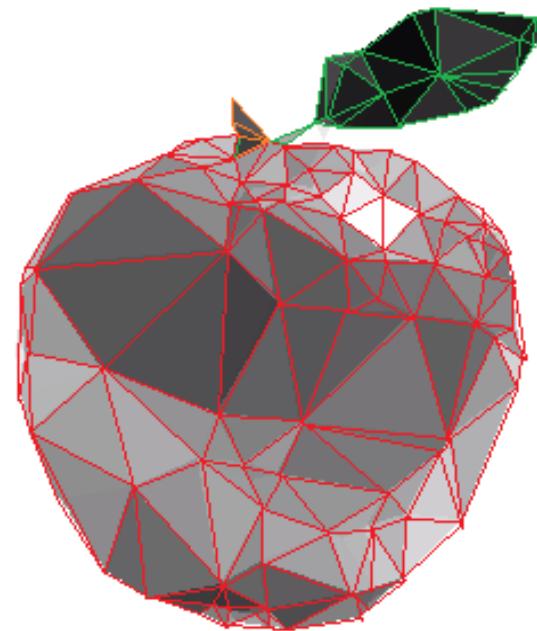
# 2次元曲面の三角分割



滑らかな曲面

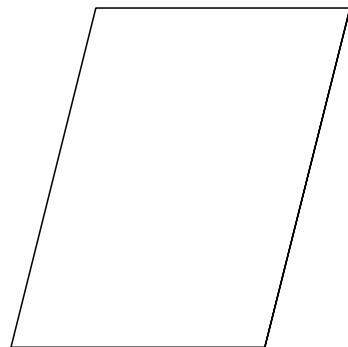
# 2次元曲面の三角分割

三角形で近似

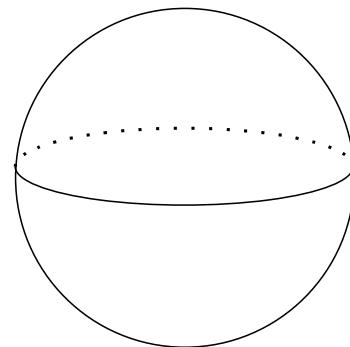


# 曲率 … 三角形の頂点に定義

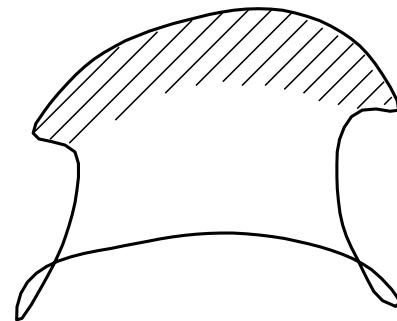
平面（曲率ゼロ）



球面（正曲率）

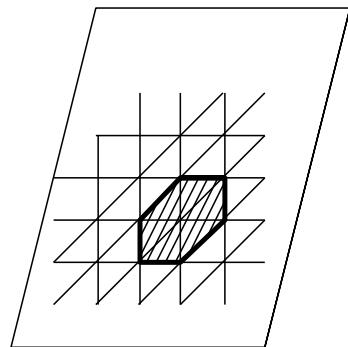


鞍状面（負曲率）

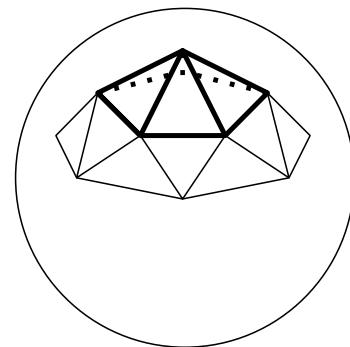


# 曲率 … 三角形の頂点に定義

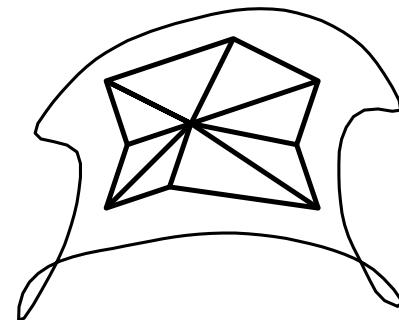
平面（曲率ゼロ）



球面（正曲率）

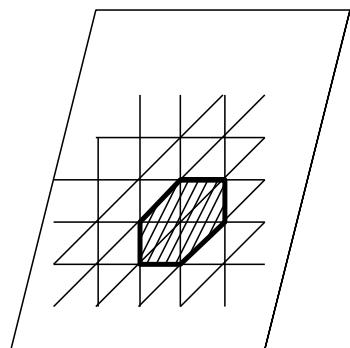


鞍状面（負曲率）

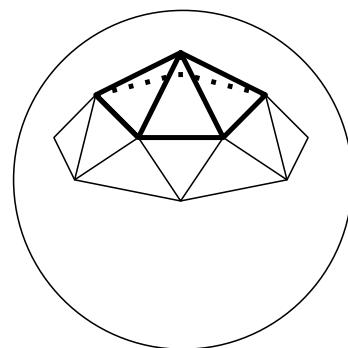


# 曲率 … 三角形の頂点に定義

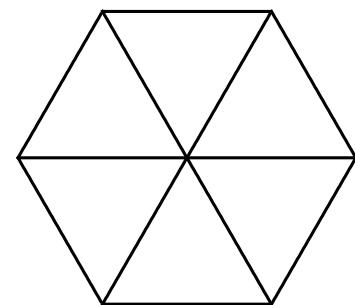
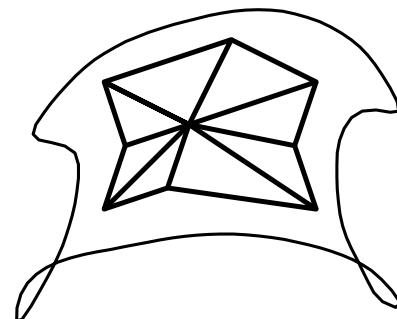
平面（曲率ゼロ）



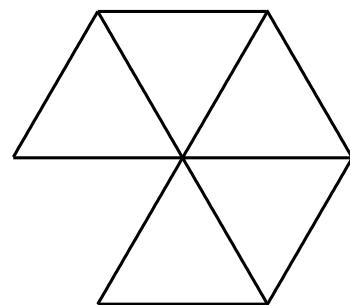
球面（正曲率）



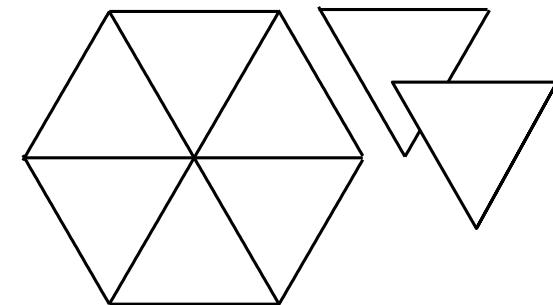
鞍状面（負曲率）



過不足なし



不足



過剰

# $n$ -単体 $\cdots n+1$ 個の頂点を持つ $n$ 次元凸面体

0-単体



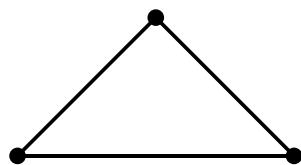
点

1-単体



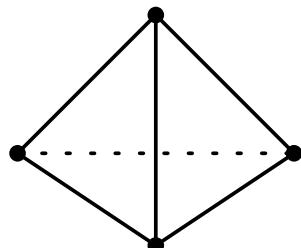
線分

2-単体



三角形

3-単体



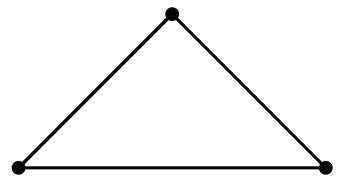
四面体

:

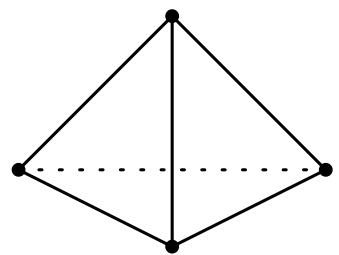
:

:

# 単体の構造

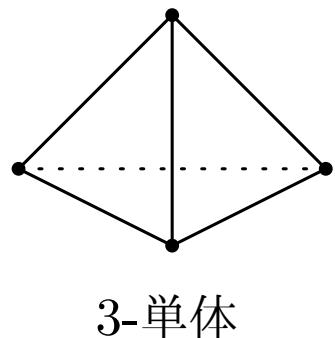
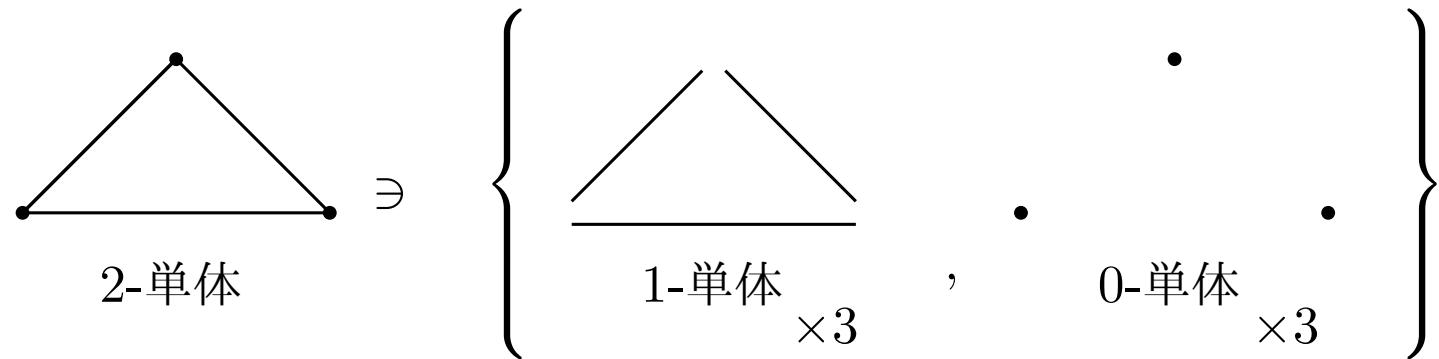


2-単体

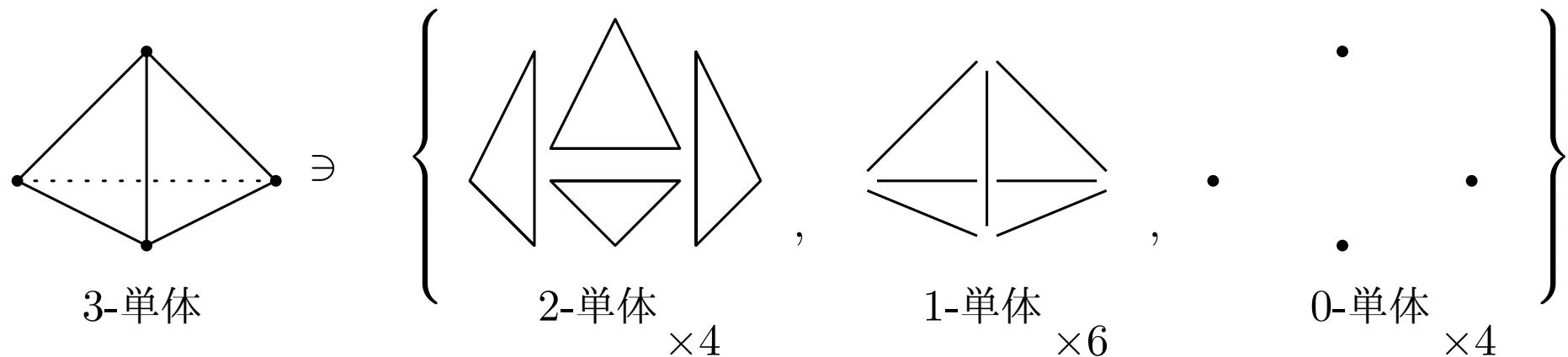
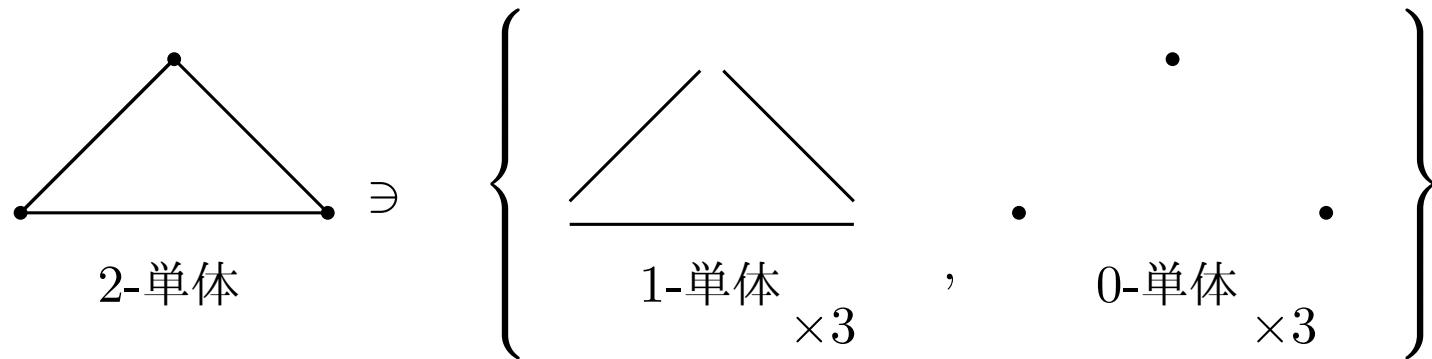


3-単体

# 単体の構造



# 単体の構造



# 単体の構造

胞の次元	形状	名称	個数
$n$	$n$ -単体	Body	${}_{n+1}C_{n+1} = 1$
$n - 1$	$(n - 1)$ -単体	Facet	${}_{n+1}C_n$
$n - 2$	$(n - 2)$ -単体	Ridge	${}_{n+1}C_{n-1}$
$n - 3$	$(n - 3)$ -単体	Peak	${}_{n+1}C_{n-2}$
$k$	$k$ -単体	$k$ 次元部分単体	${}_{n+1}C_{k+1}$
3	3-単体	3 次元胞	${}_{n+1}C_4$
2	2-単体	面	${}_{n+1}C_3$
1	1-単体	辺	${}_{n+1}C_2$
0	0-単体	頂点	${}_{n+1}C_1$

表 1  $n$ -単体の胞の次元, 形状, 名称, 個数

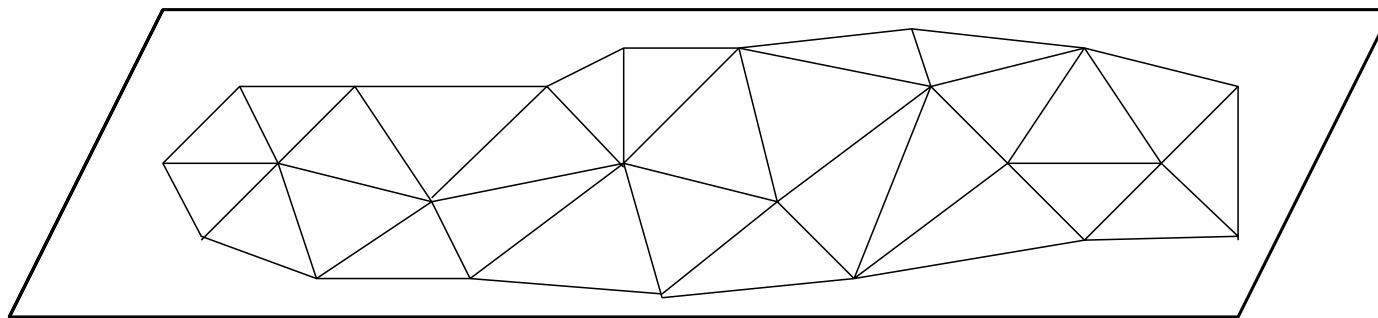
${}_kC_j$  は二項係数:  ${}_kC_j = \frac{k!}{j!(k-j)!}.$

$n + 1$  個ある  $n$ -単体の頂点から  $k + 1$  個選ぶと  $k$ -単体が組める.

# Delaunay 格子と Voronoi ポリゴン

単体分割によって作られた格子

… Delaunay 格子



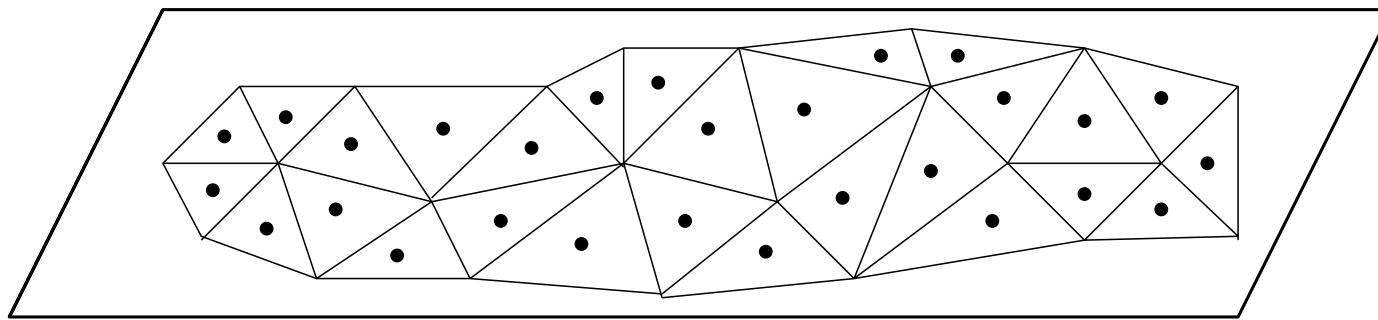
1. 各单体の外心を求める
2. 隣り合う外心を結ぶ

… Voronoi ポリゴン

# Delaunay 格子と Voronoi ポリゴン

単体分割によって作られた格子

… Delaunay 格子



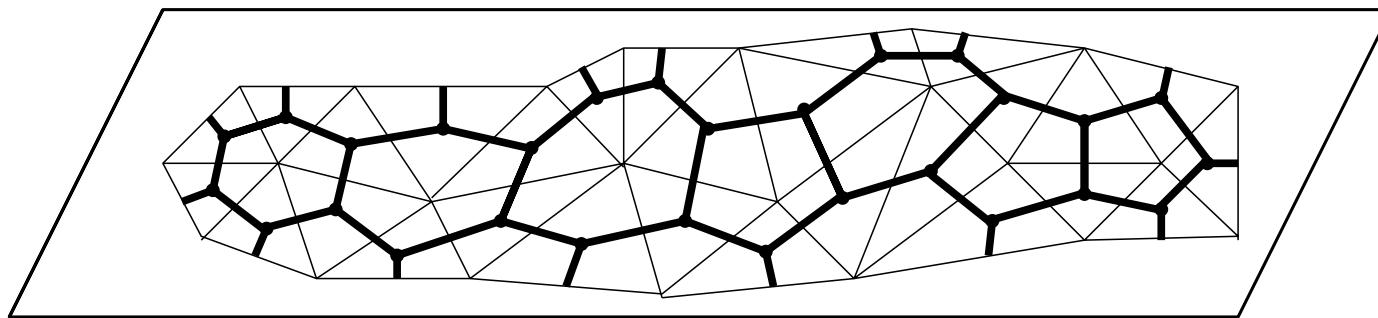
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# Delaunay 格子と Voronoi ポリゴン

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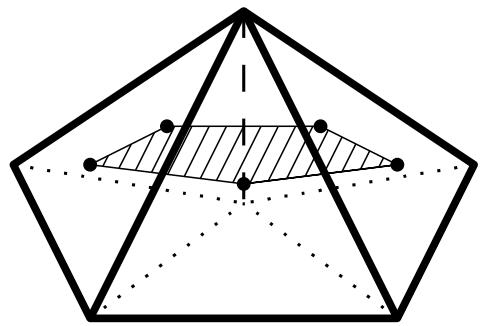
… Delaunay 格子



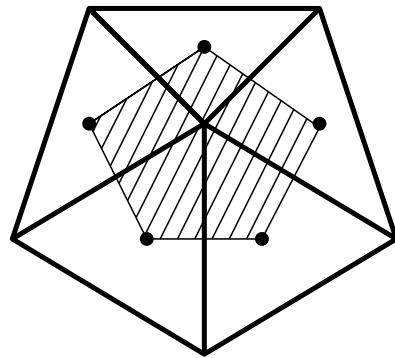
1. 各单体の外心を求める
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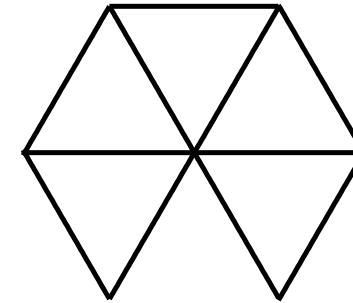
# Hinge と Voronoi ポリゴンと Deficit Angle



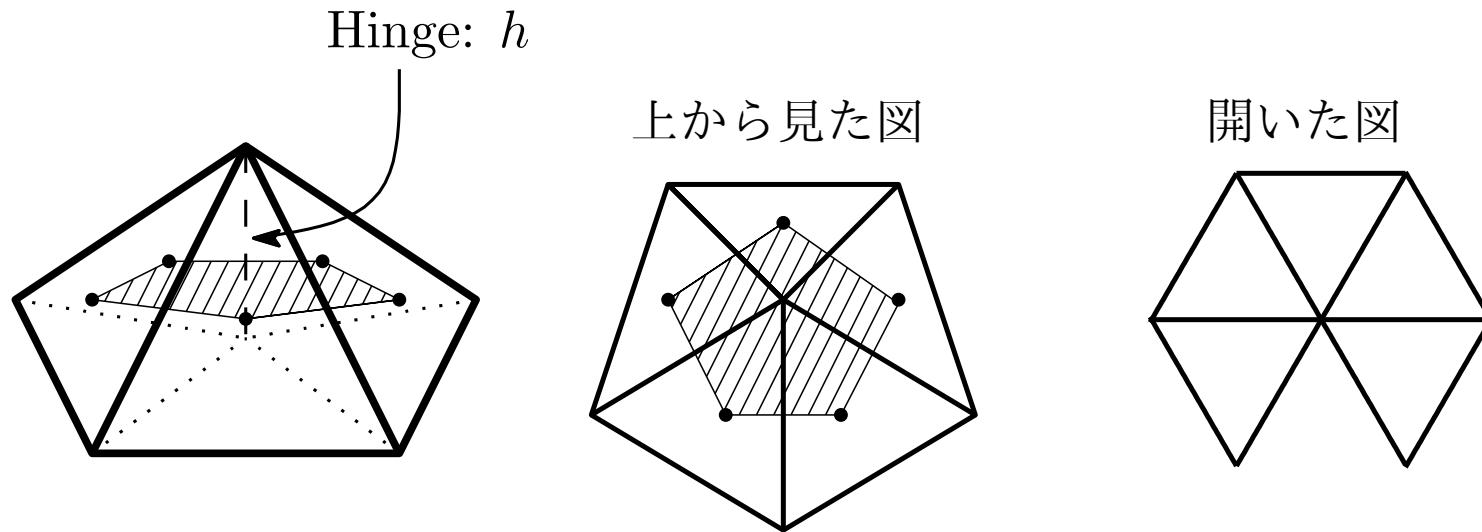
上から見た図



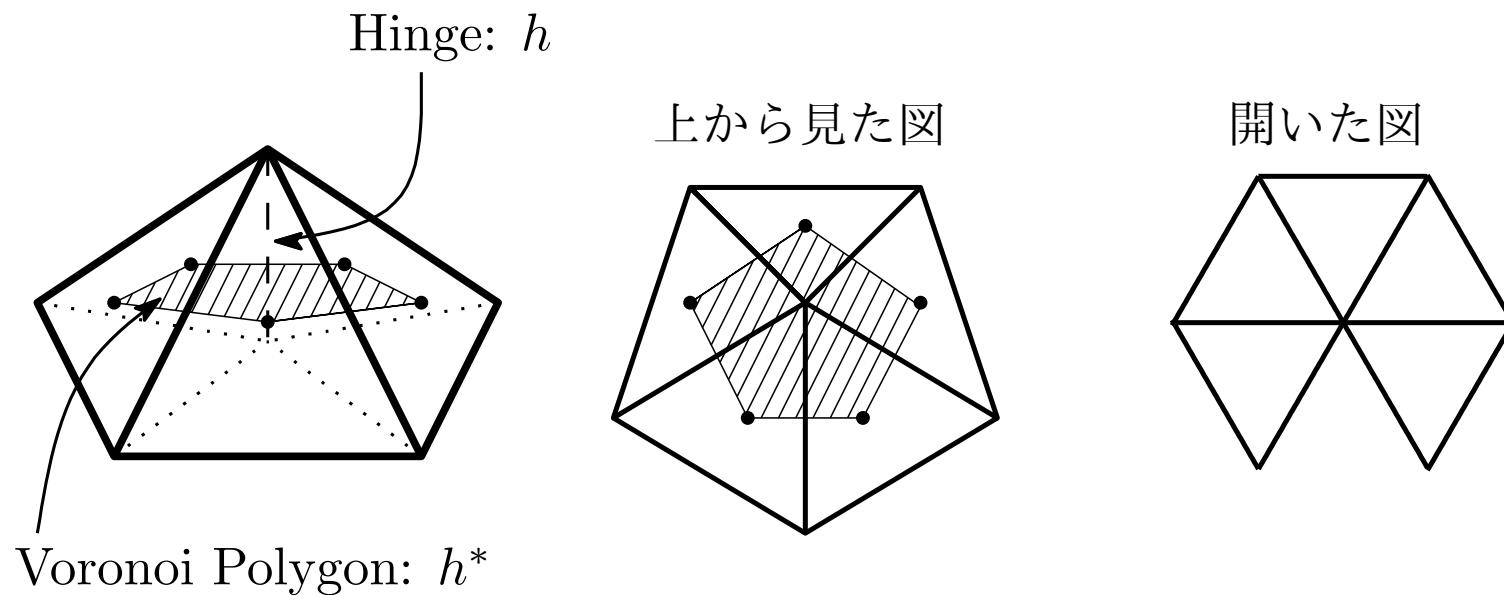
開いた図



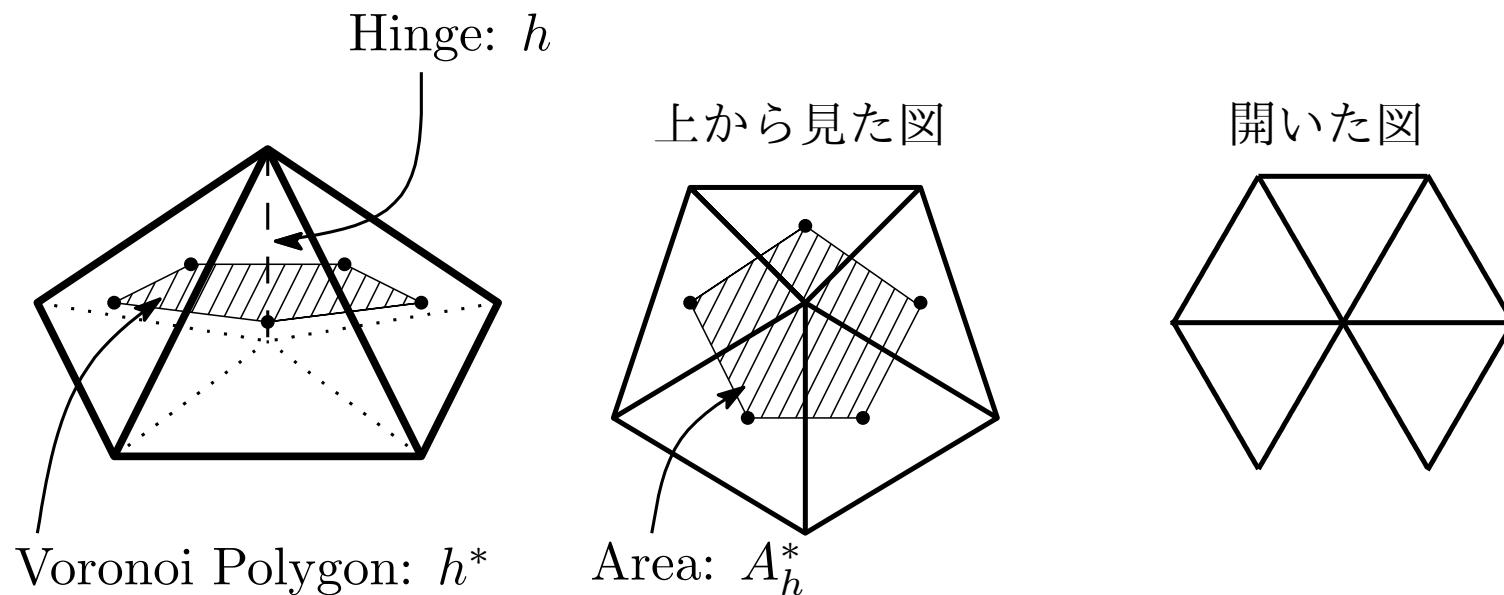
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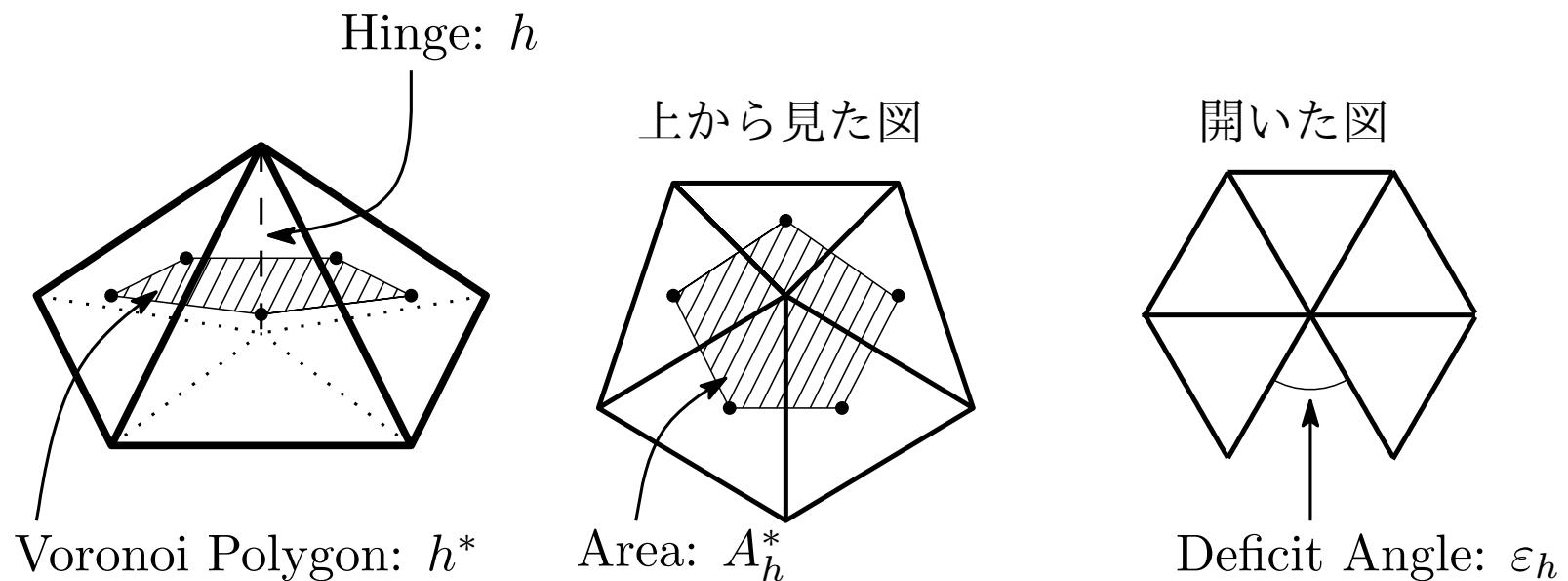
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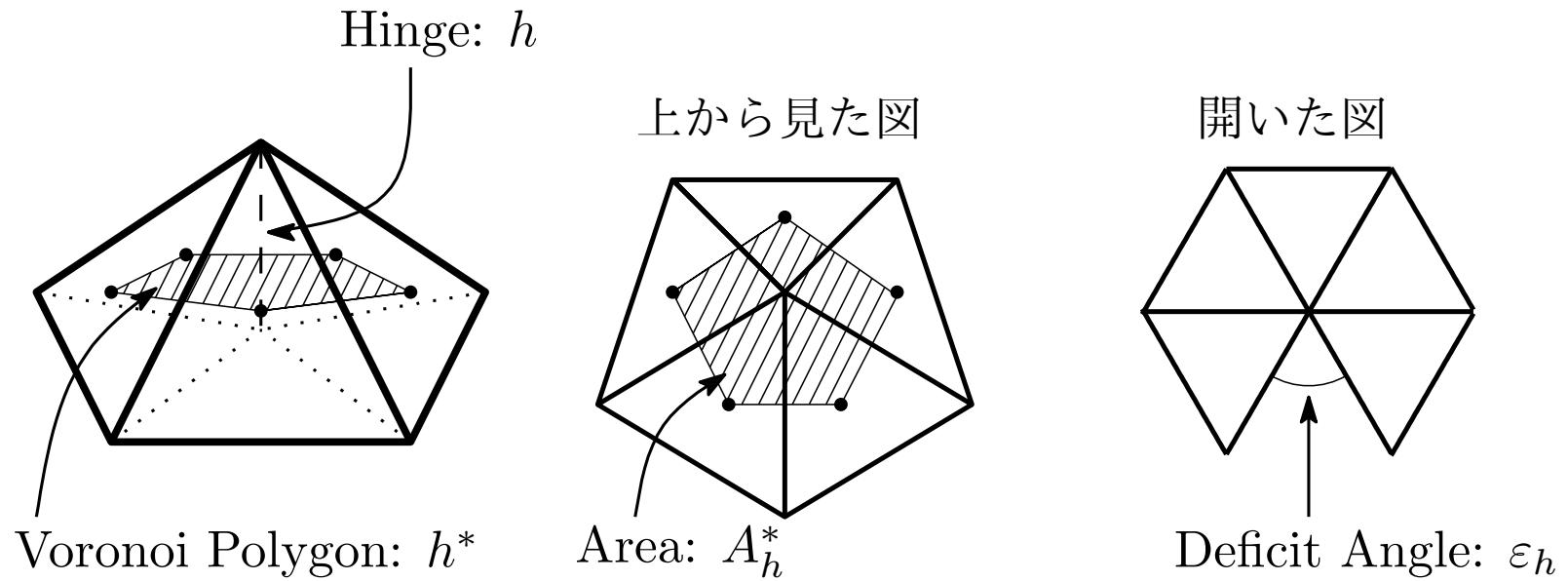
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# Hinge と Voronoi ポリゴンと Deficit Angle



# Hinge と Voronoi ポリゴンと Deficit Angle



$h$  と  $h^*$  と  $\varepsilon_h$ : 1 対 1 対 1 対応

# Regge Calculus とは

(T. Regge, Il Nuovo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる
2. 単体の配置を変化させる

# Regge Calculus とは (T. Regge, Il Nuovo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる  
→ Regge Calculus
2. 単体の配置を変化させる

# Regge Calculus とは (T. Regge, Il Nuovo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

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→ Regge Calculus

2. 単体の配置を変化させる

→ Dynamical Triangulation

# Regge Calculus とは (T. Regge, Il Nuovo Cim. **19**, 558 (1961))

多様体の Dynamics を単体分割で表現するには

1. 各単体の形状を変化させる

→ Regge Calculus

2. 単体の配置を変化させる

→ Dynamical Triangulation

## 2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Crav. **14**, 199 (1997))

- Ricci Scalar Curvature
- Proper Volume Element
- Hilbert Action

# Ricci Scalar Curvature の変形

$$R = g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$$

# Ricci Scalar Curvature の変形

$$R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$$

$$R = D(D - 1)K$$

# Ricci Scalar Curvature の変形

$$R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$$

$$R = D(D - 1)K$$

Dimension

# Ricci Scalar Curvature の変形

$$R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$$

$$R = D(D - 1)K$$

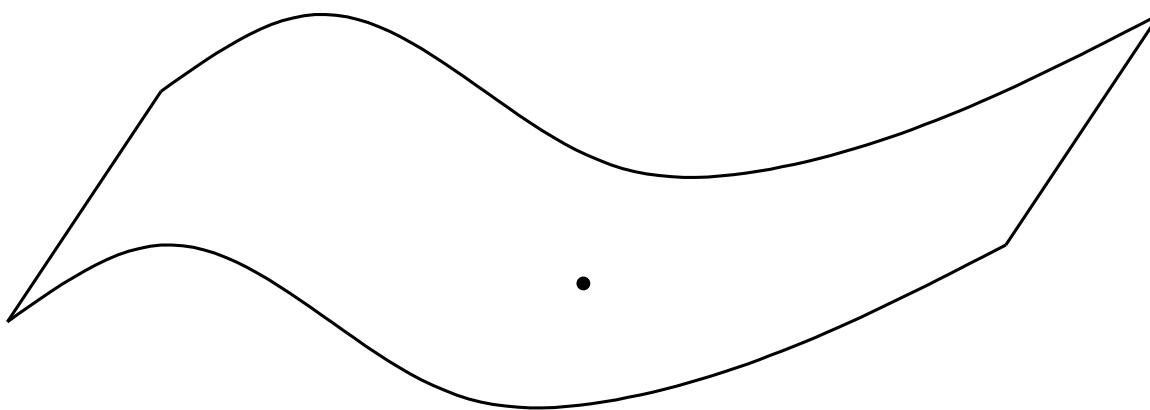
Dimension

Gaussian Curvature

```
graph TD; R["R = D(D - 1)K"] --> Dimension["Dimension"]; R --> GaussianCurvature["Gaussian Curvature"]
```

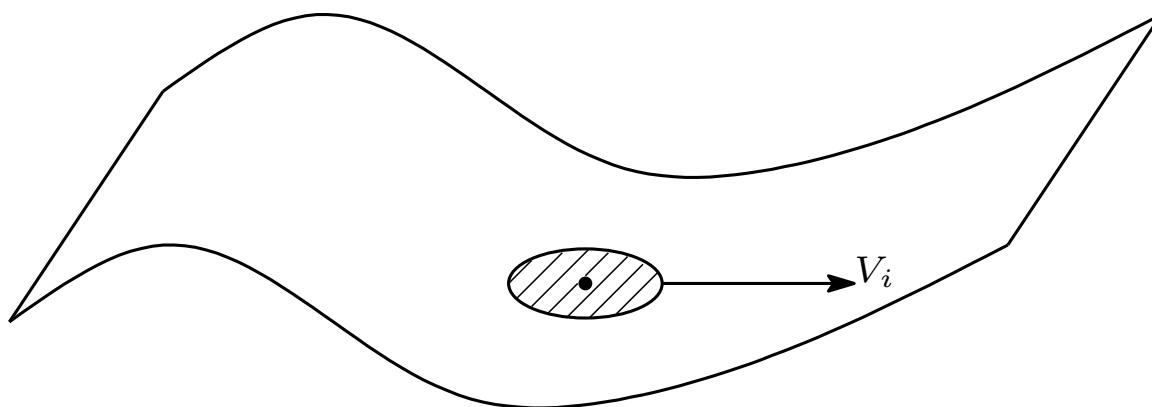
# Gaussian Curvature

$K$



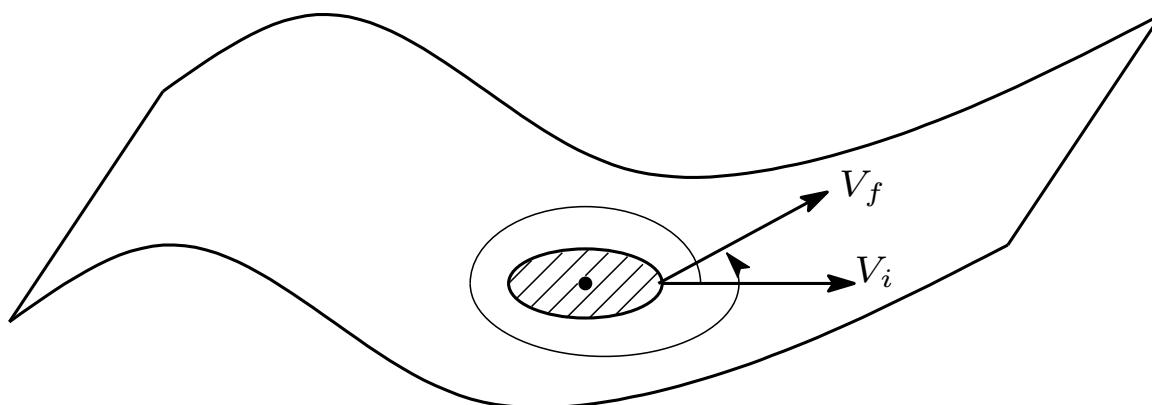
# Gaussian Curvature

$K$



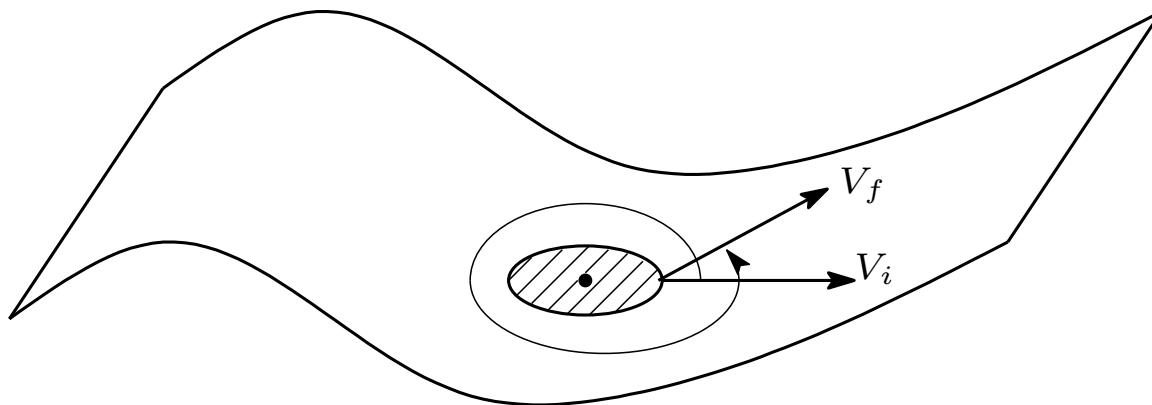
# Gaussian Curvature

$K$



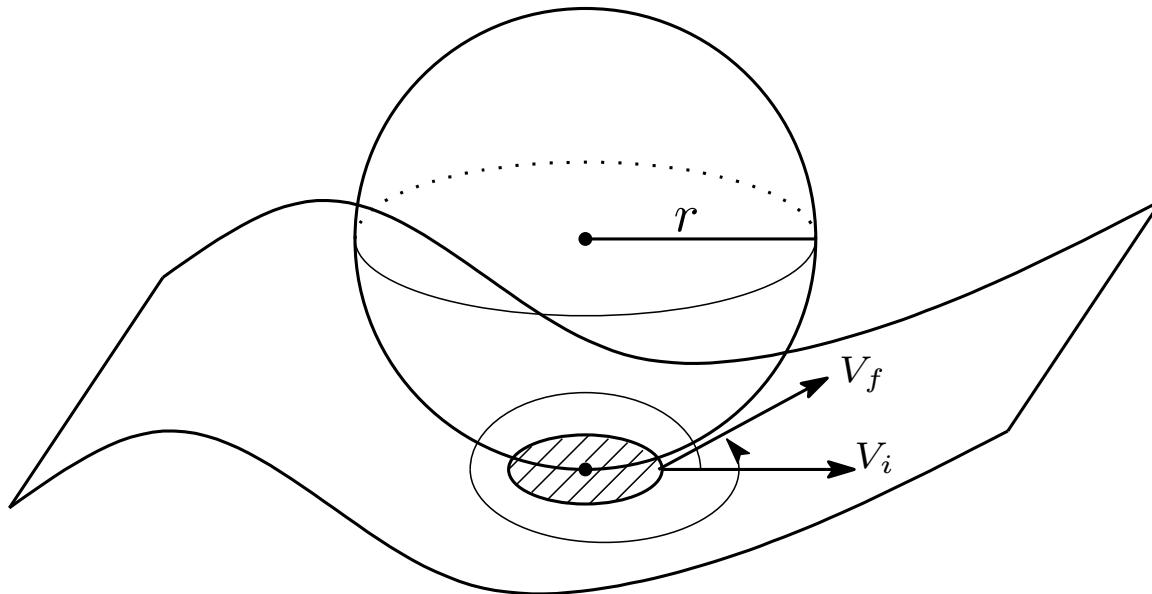
# Gaussian Curvature

$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}}$$



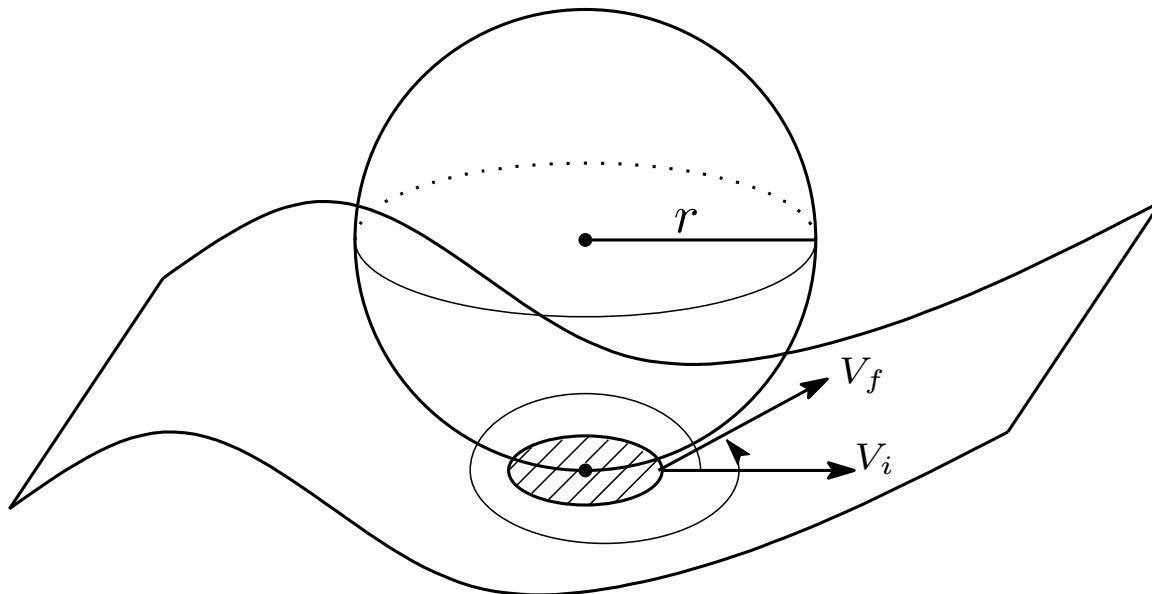
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$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}}$$

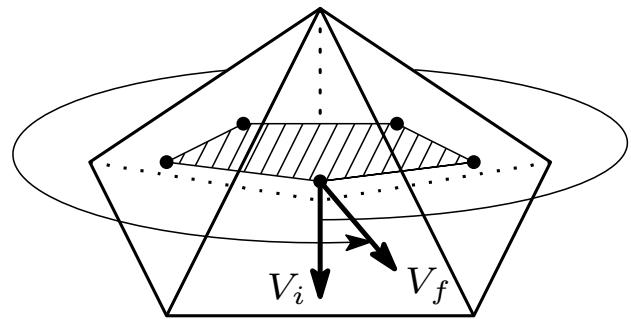


# Gaussian Curvature

$$K = \lim_{\text{Area} \rightarrow 0} \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}} = \frac{1}{r^2}$$

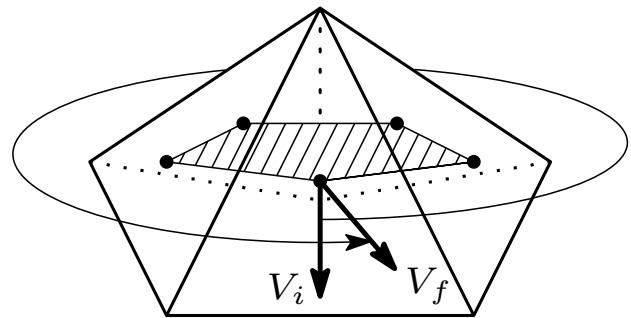


# Gaussian Curvature

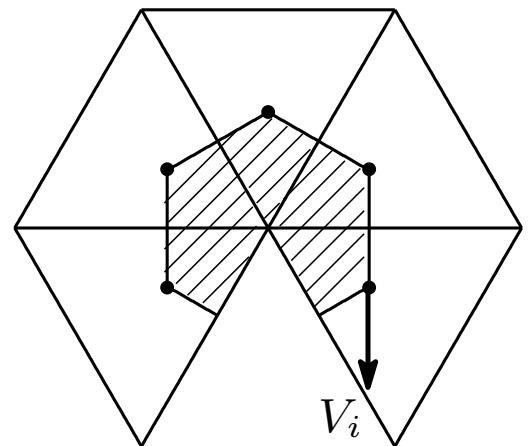


Area Circumnavigated  
 $= A_h^*$

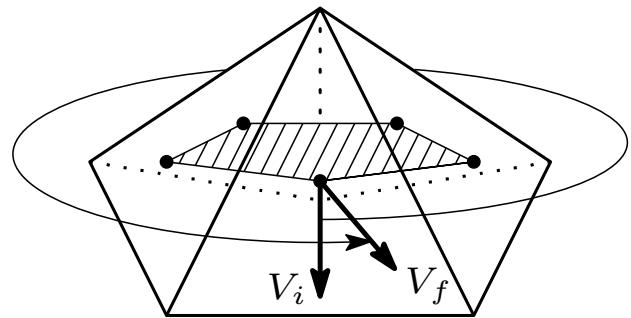
# Gaussian Curvature



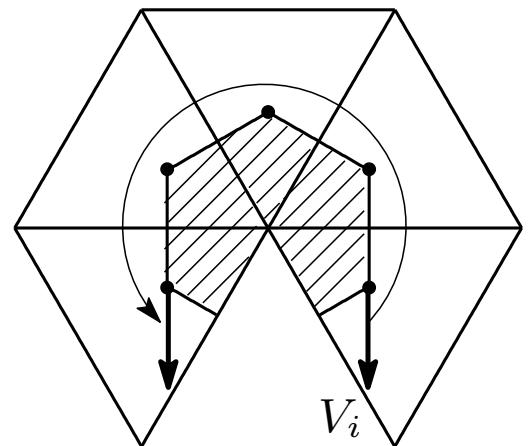
Area Circumnavigated  
=  $A_h^*$



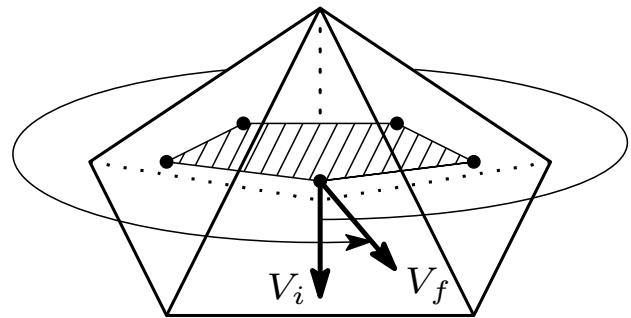
# Gaussian Curvature



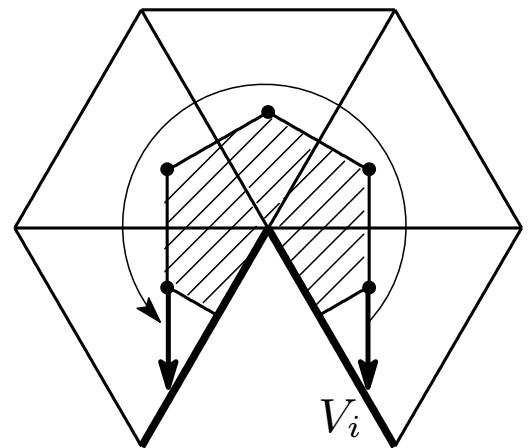
Area Circumnavigated  
=  $A_h^*$



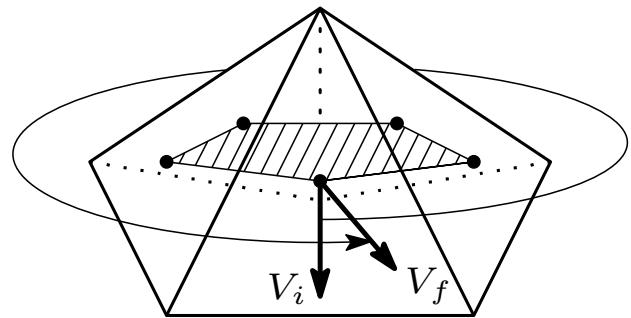
# Gaussian Curvature



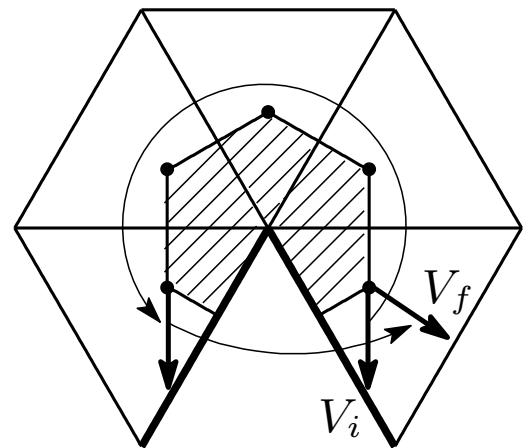
Area Circumnavigated  
 $= A_h^*$



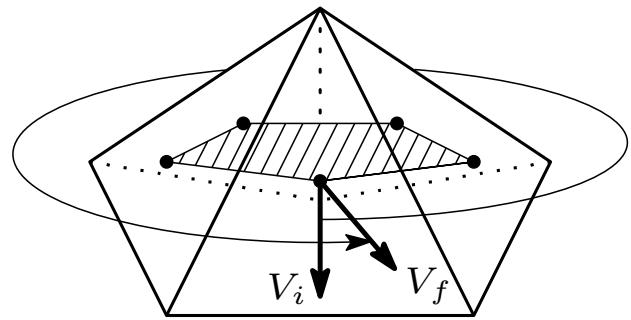
# Gaussian Curvature



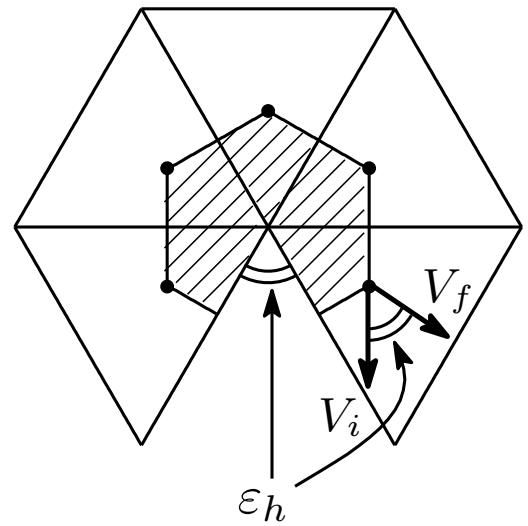
Area Circumnavigated  
=  $A_h^*$



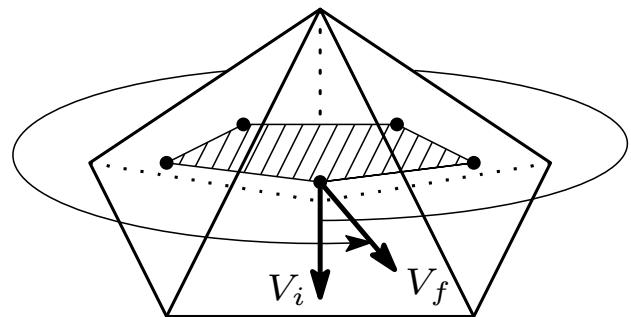
# Gaussian Curvature



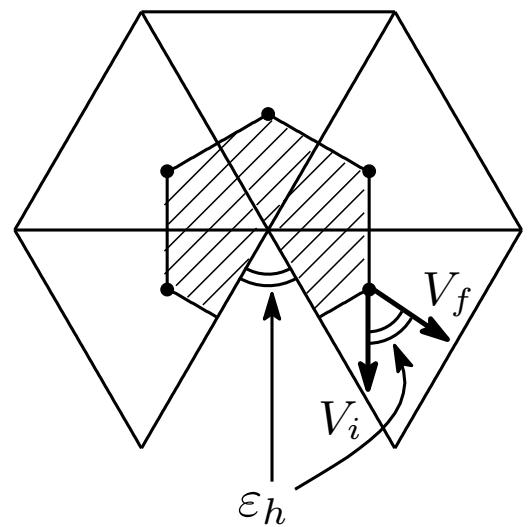
Area Circumnavigated  
=  $A_h^*$



# Gaussian Curvature



Area Circumnavigated  
=  $A_h^*$



Angle that Vector  
is Rotated  
=  $\varepsilon_h$

# Ricci Scalar Curvature in Regge Calculus

$$K_{\text{Regge}} = \frac{\varepsilon_h}{A_h^*}$$

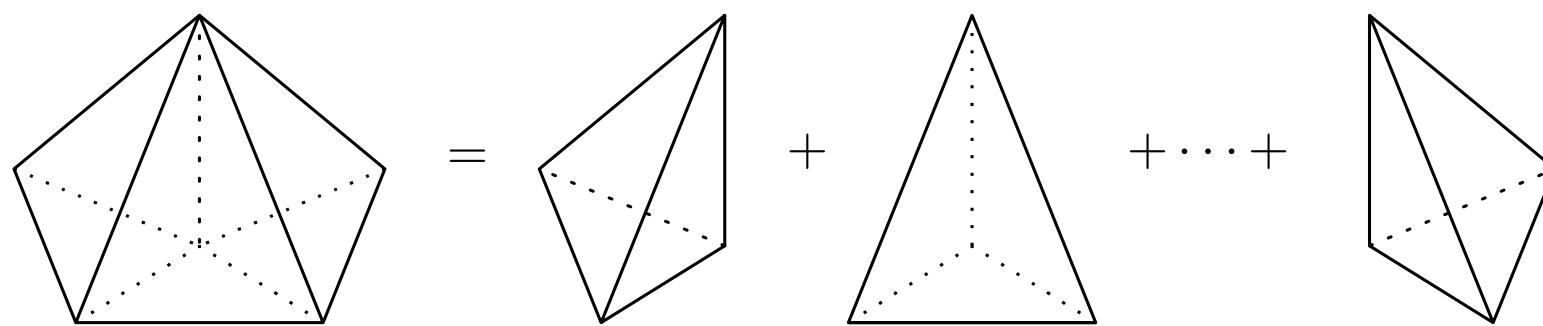
$$R_{\text{Regge}} = D(D-1) \frac{\varepsilon_h}{A_h^*}$$

## 2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Crav. **14**, 199 (1997))

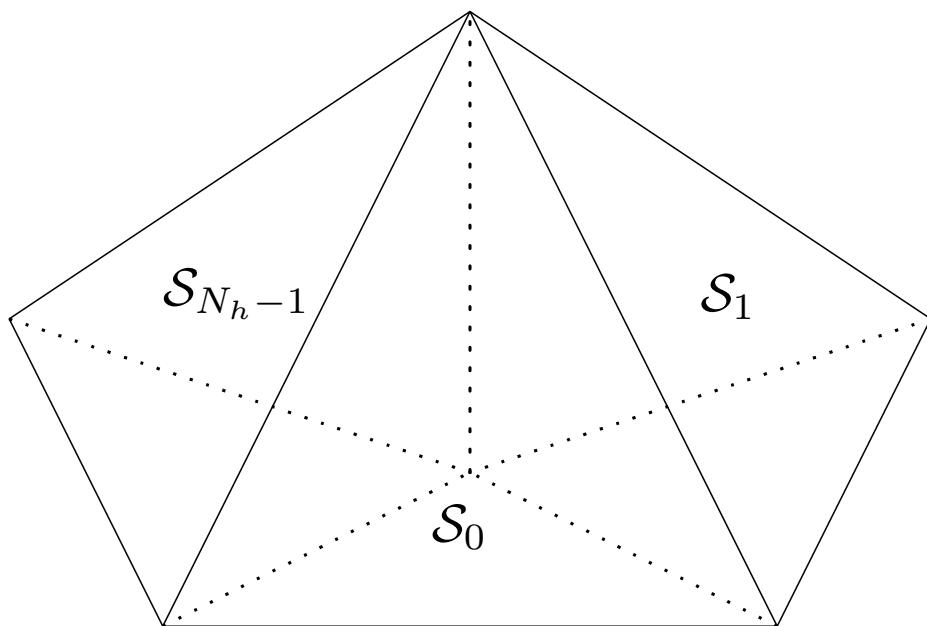
- Ricci Scalar Curvature
- **Proper Volume Element**
- Hilbert Action

複体 … 有限個の単体からなる立体



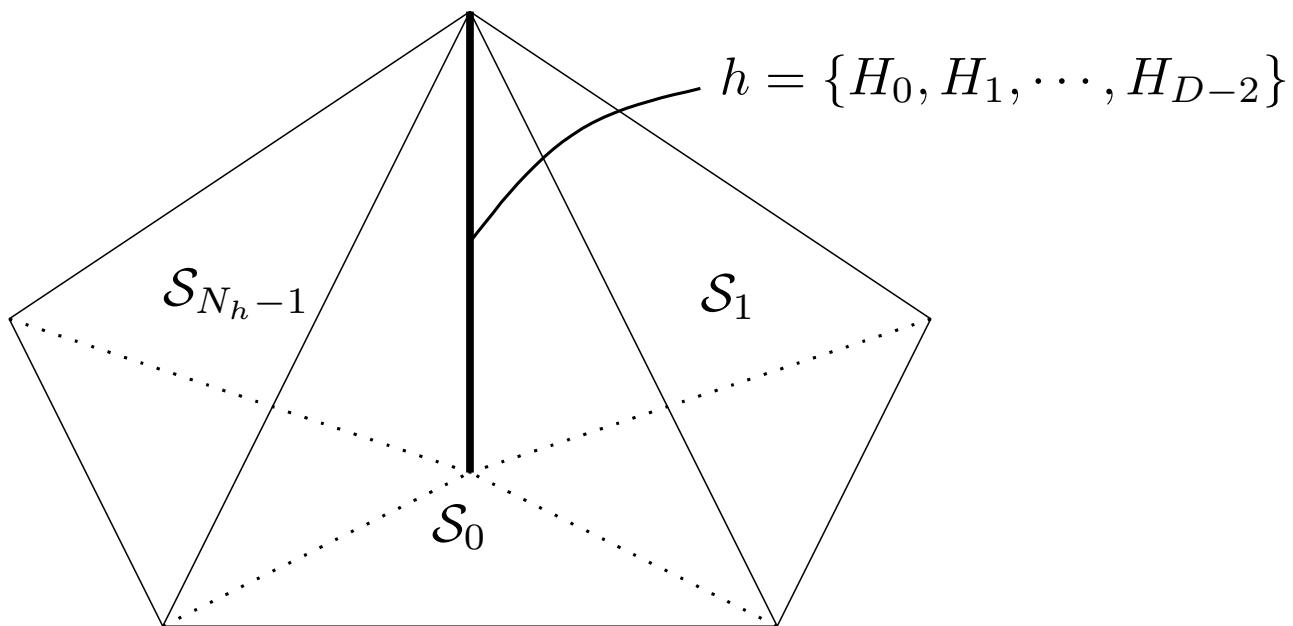
# 複体の構造

$N_h$  個の単体  $\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{N_h-1}\}$  から成る  $D$  次元複体



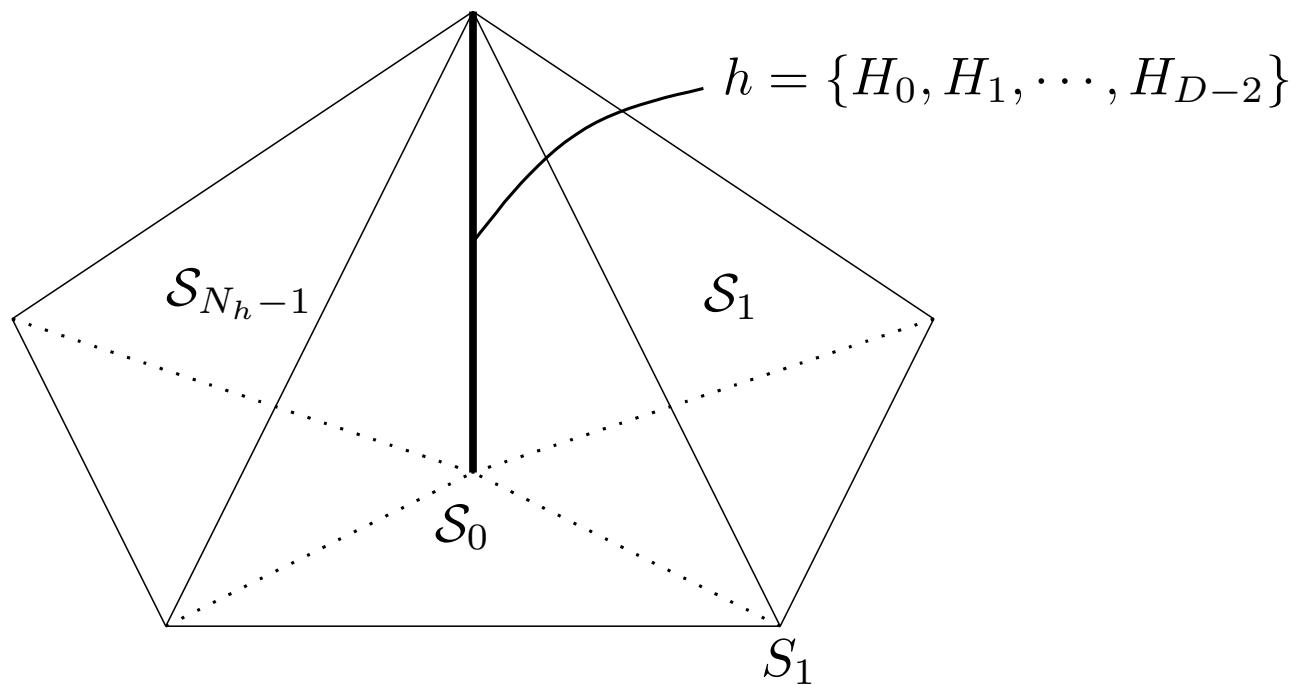
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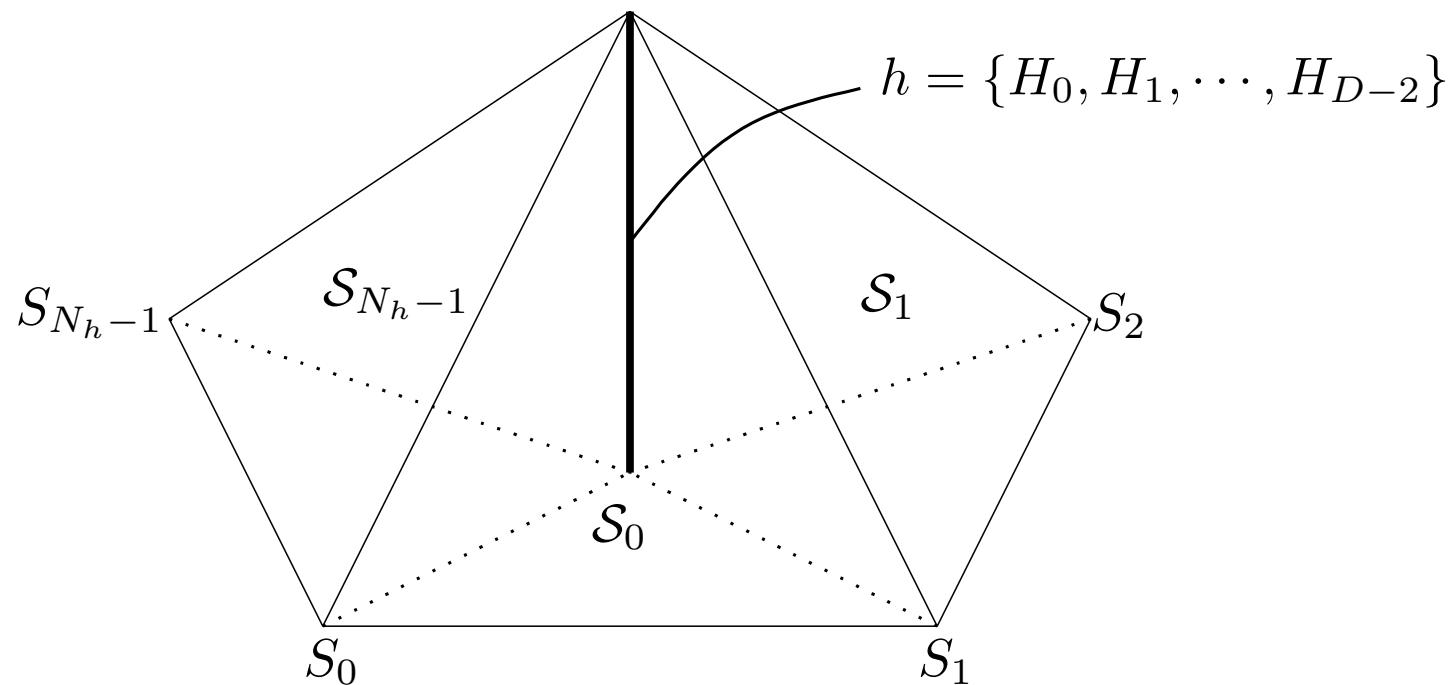
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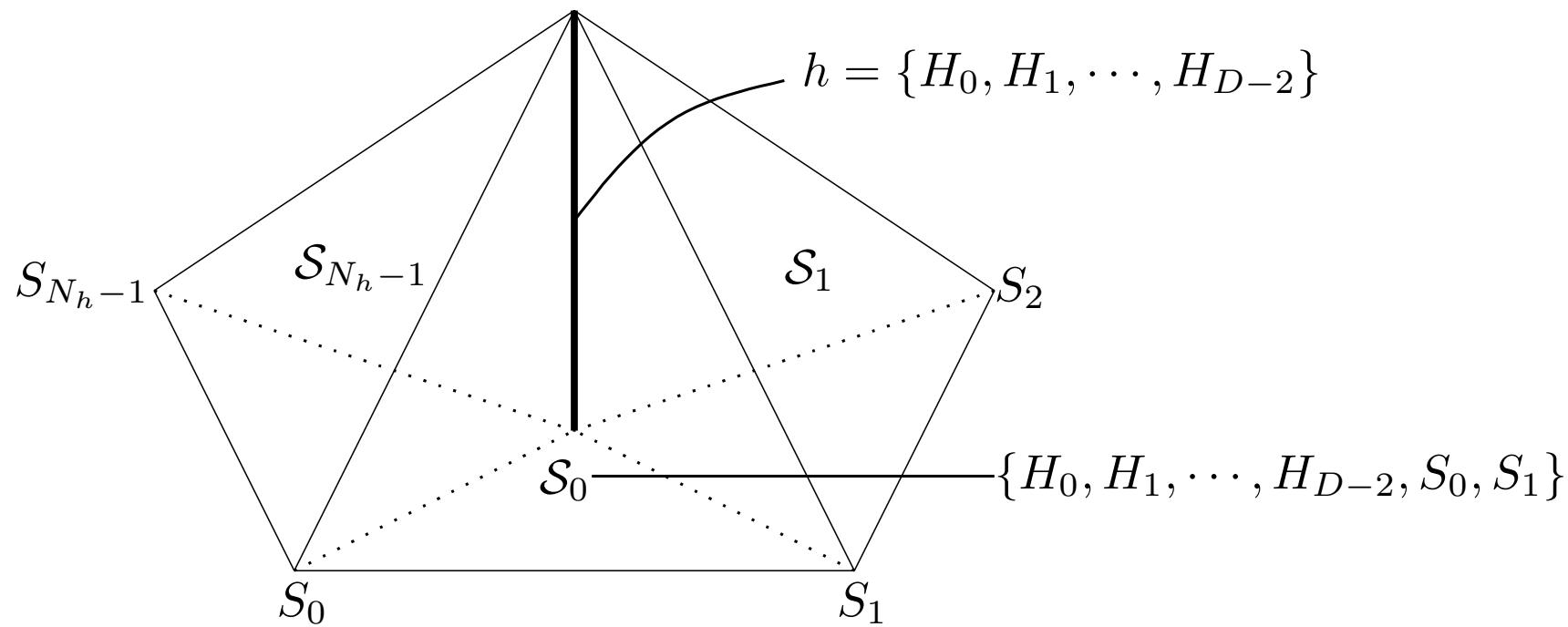
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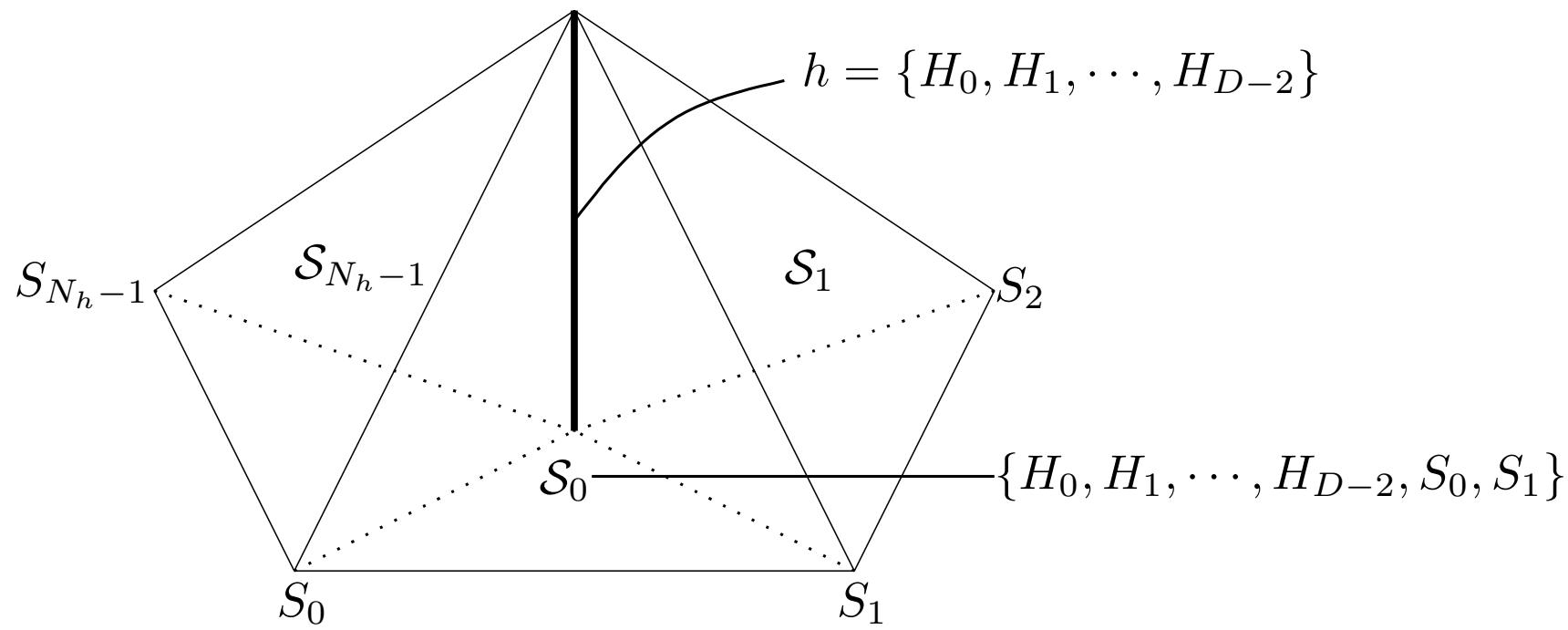
# 複体の構造

$N_h$  個の単体  $\{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{N_h-1}\}$  から成る  $D$  次元複体



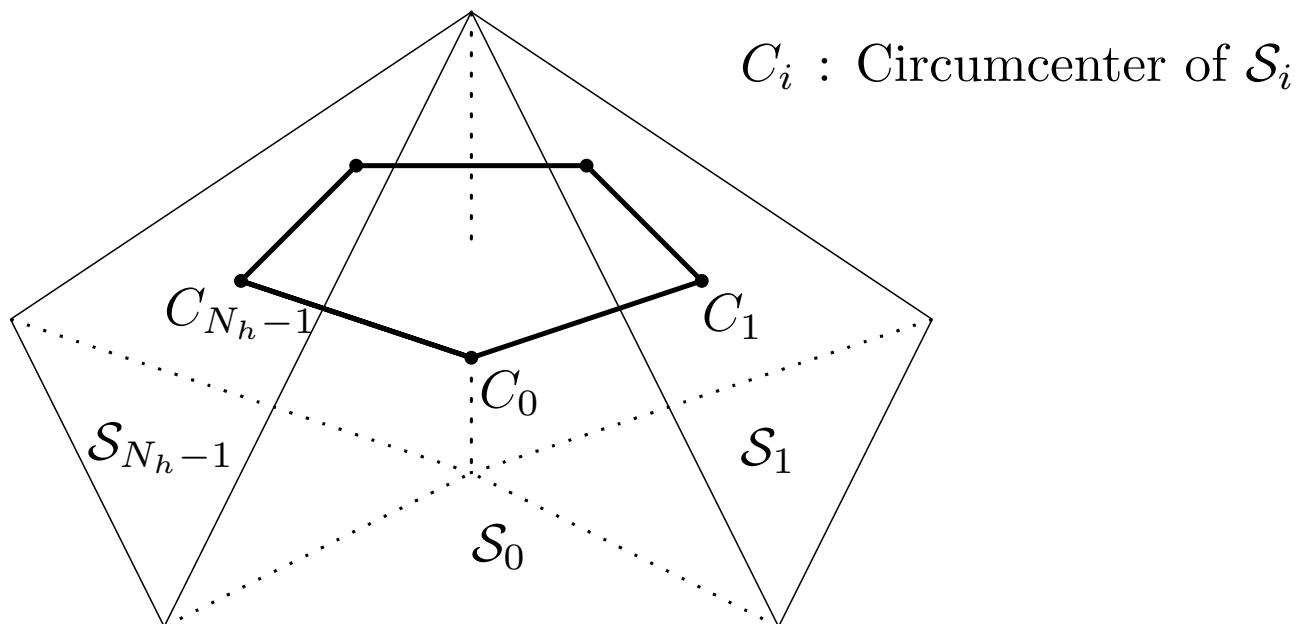
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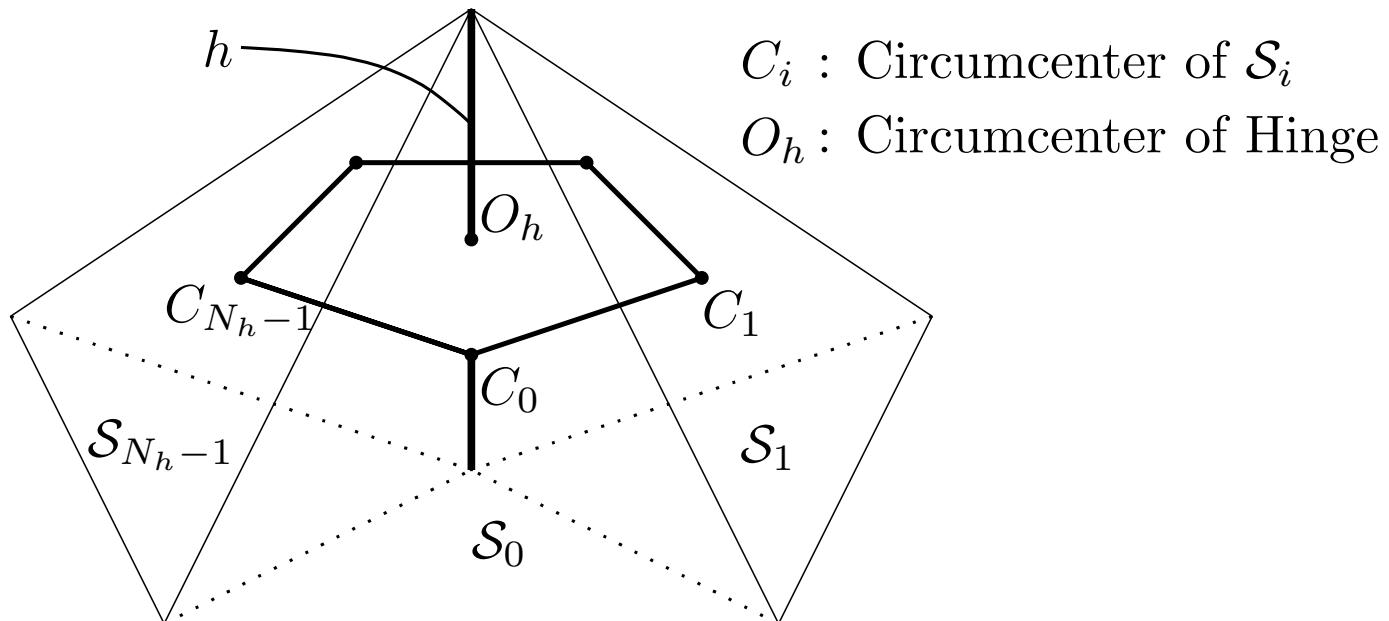
$$\mathcal{S}_i = \{H_0, H_1, \dots, H_{D-2}, S_i, S_{i+1 \bmod N_h}\}$$

# Voronoi ポリゴンの分解



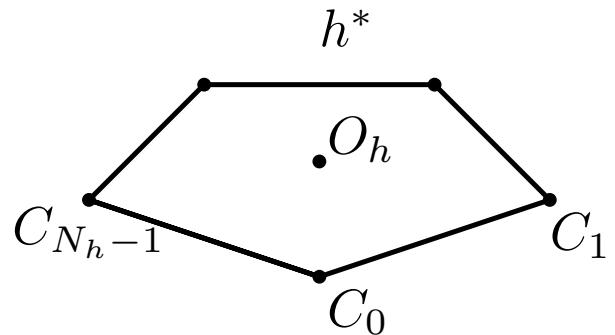
$$h^* = \{C_0, C_1, \dots, C_{N_h-1}\}$$

# Voronoi ポリゴンの分解

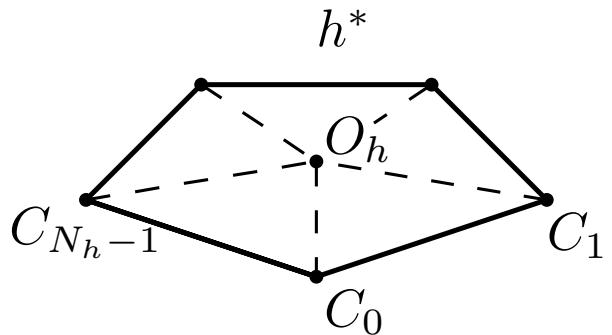


$$h^* = \{C_0, C_1, \dots, C_{N_h-1}\}$$

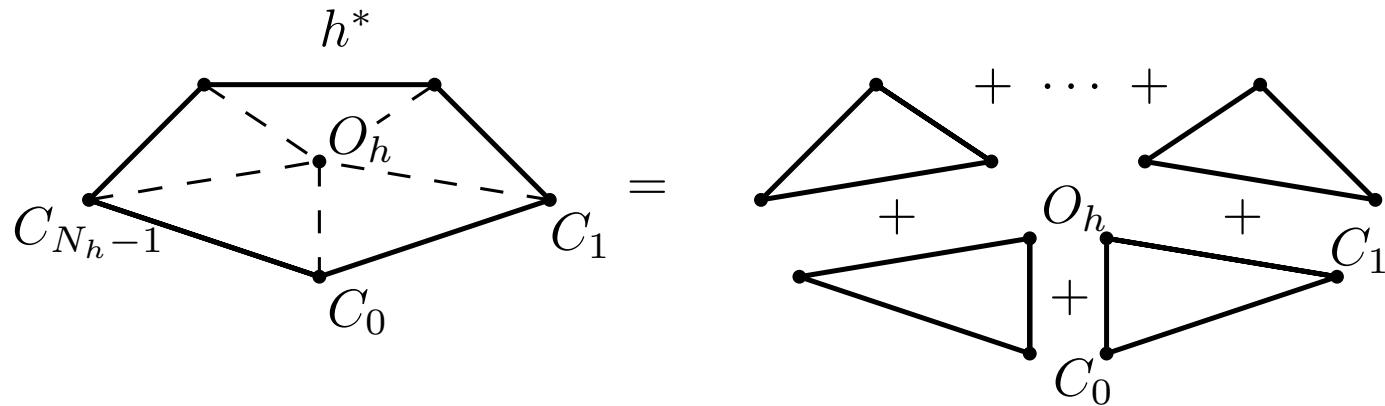
# Voronoi ポリゴンの分解



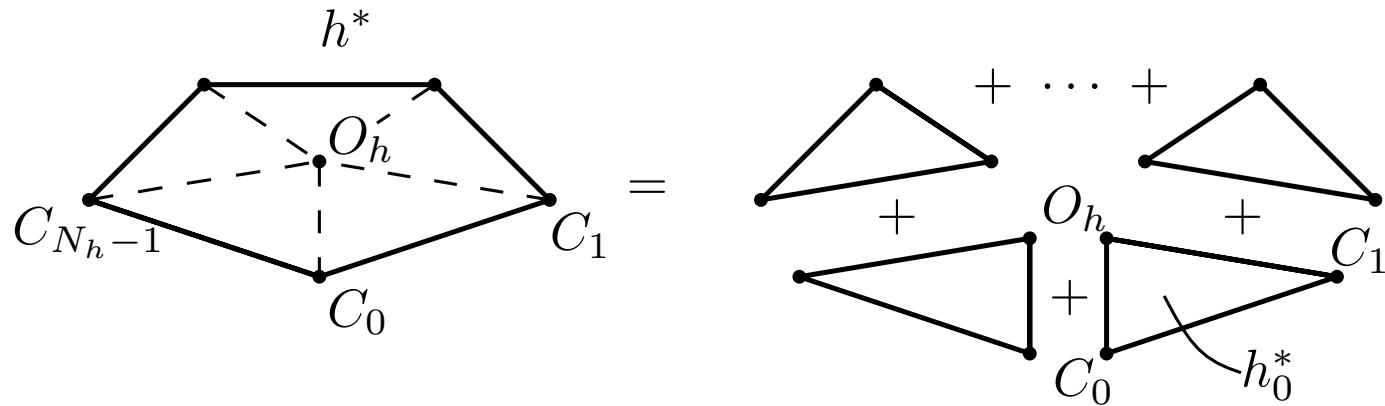
# Voronoi ポリゴンの分解



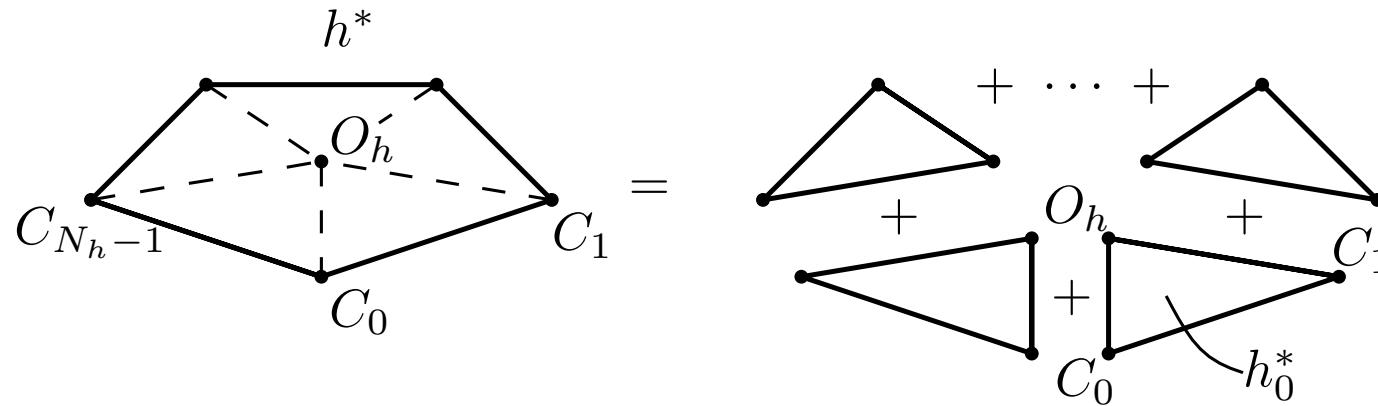
# Voronoi ポリゴンの分解



# Voronoi ポリゴンの分解



# Voronoi ポリゴンの分解

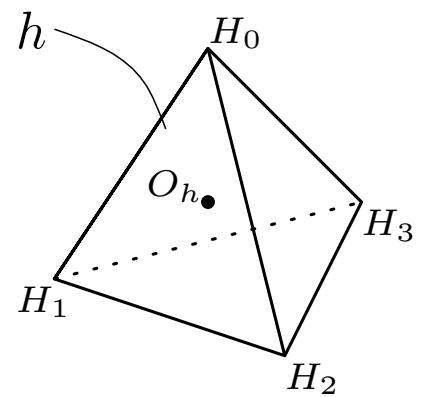


$$h_i^* = \{O_h, C_i, C_{i+1 \bmod N_h}\}$$

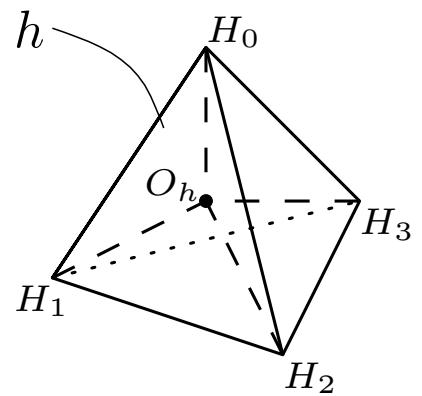
$A_{h_i}^*$ : Area of  $h_i^*$

$$A_h^* = \sum_{i=0}^{N_h-1} A_{h_i}^*$$

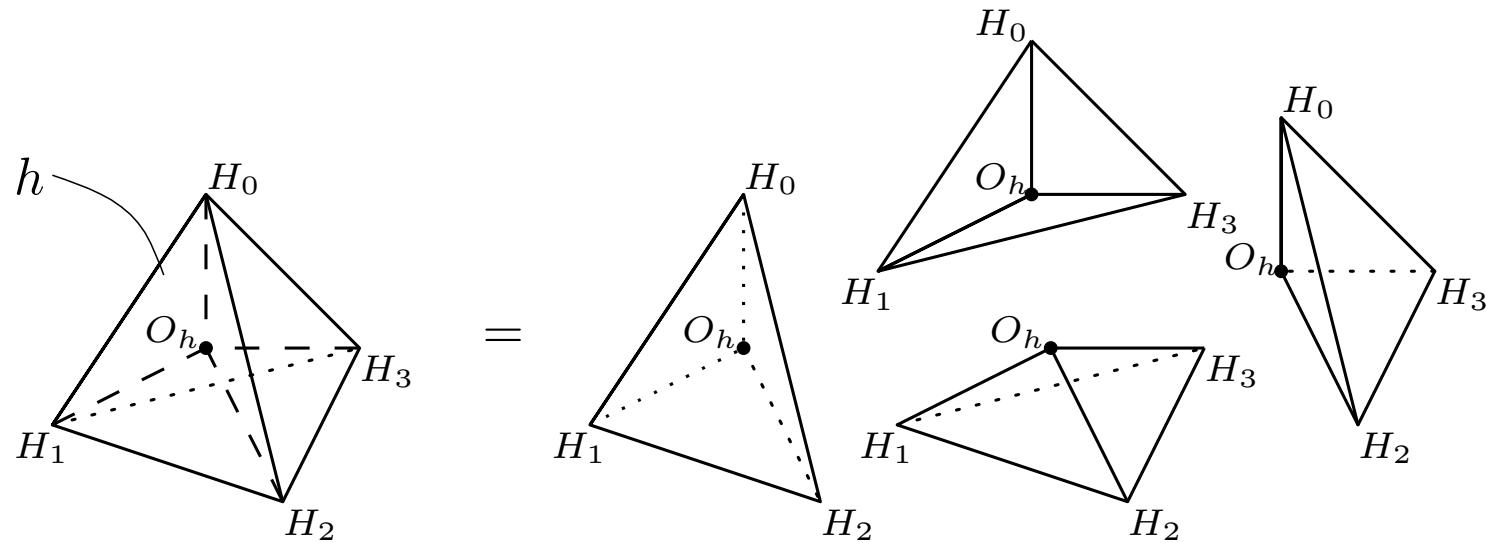
# Hinge の分解



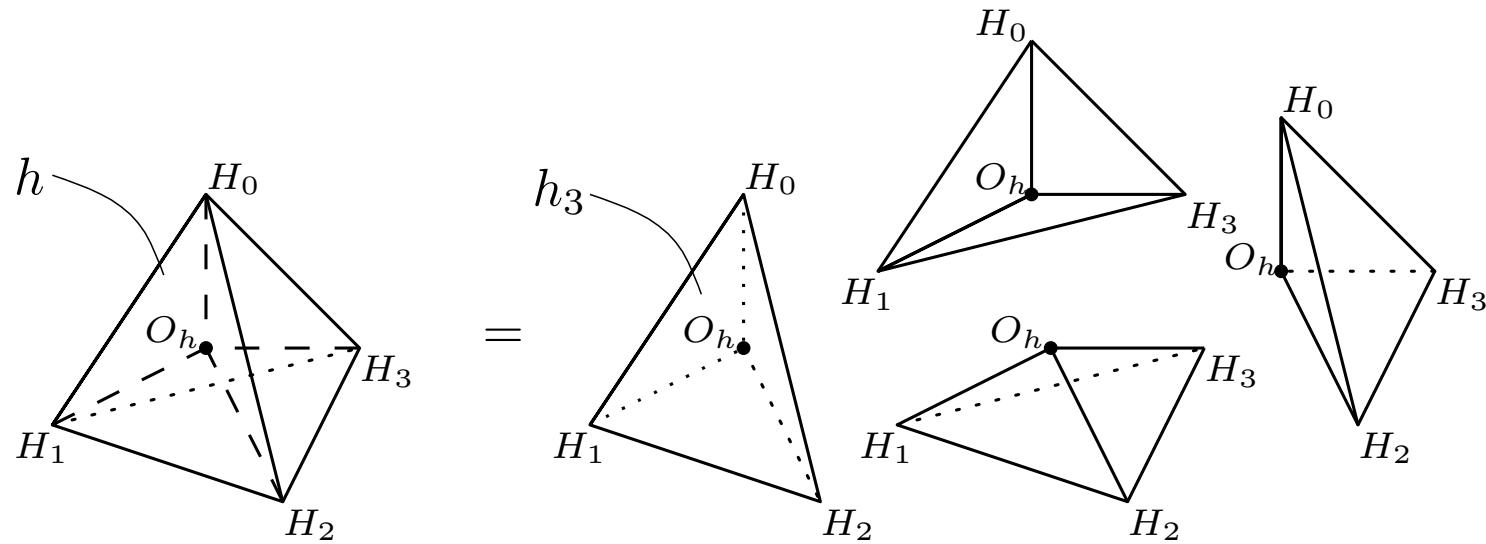
# Hinge の分解



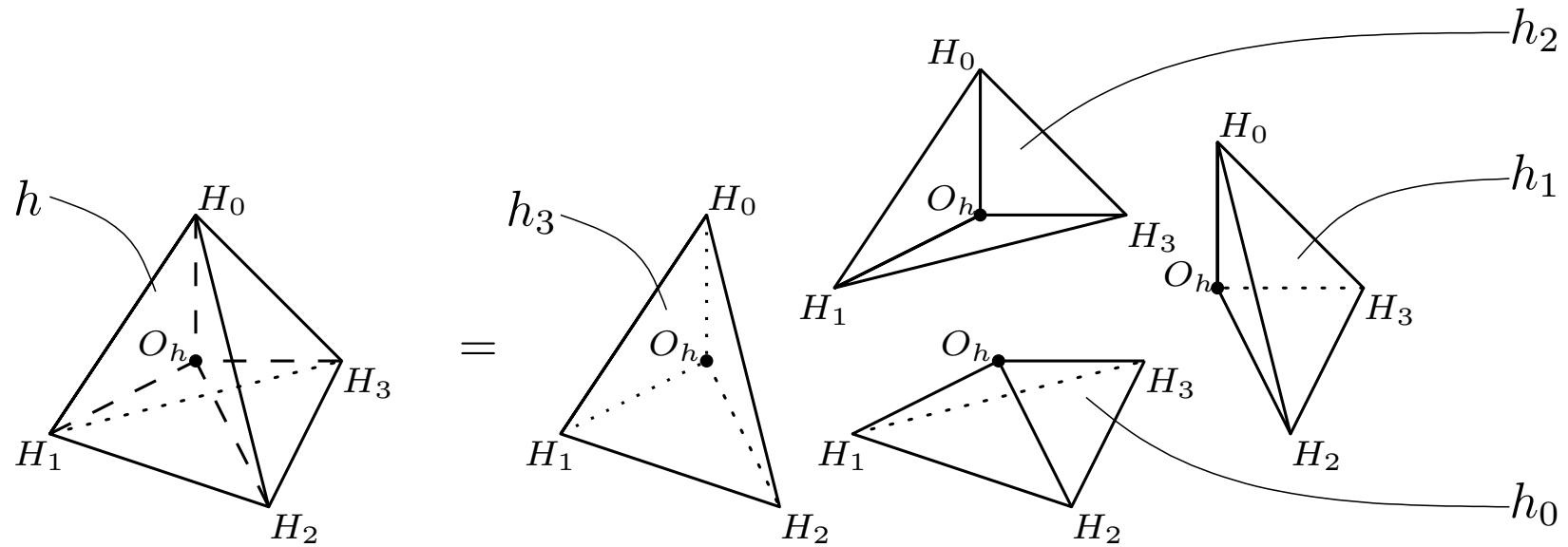
# Hinge の分解



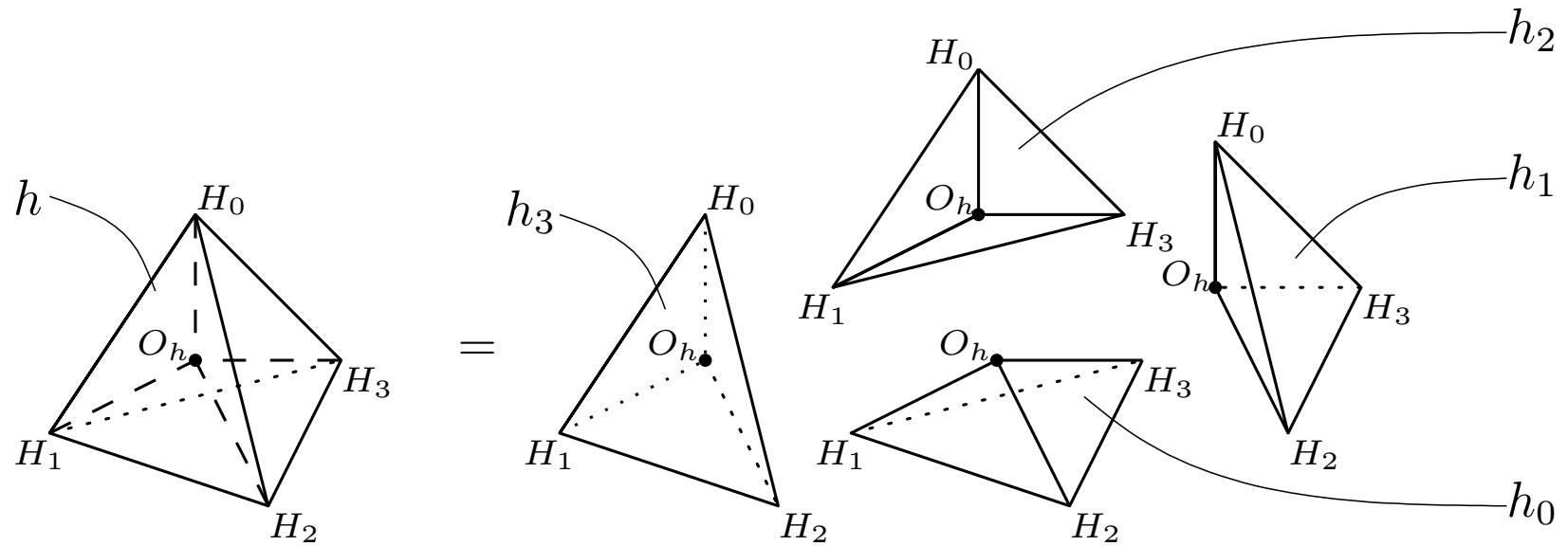
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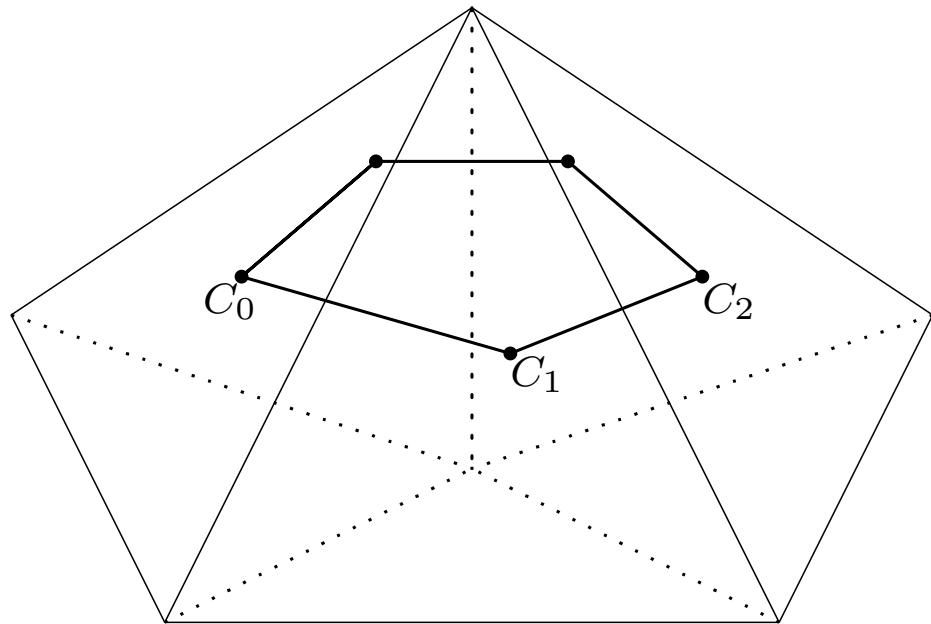


$$h_i = \{O_h, H_0, \dots, H_{i-1}, \wedge, H_{i+1}, \dots, H_{D-2}\}$$

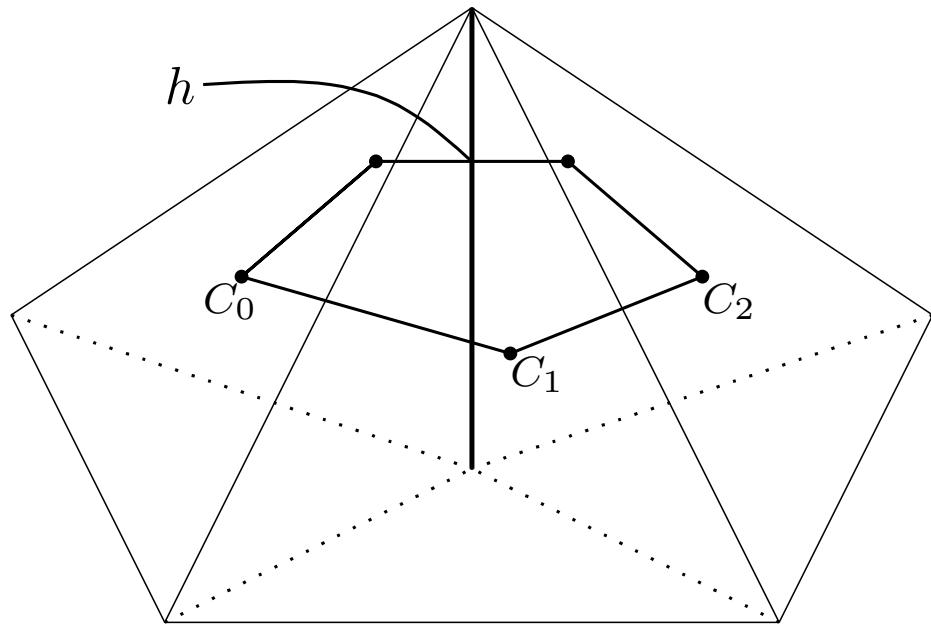
$A_{h_i}$ : Volume of  $h_i$

$$A_h = \sum_{i=0}^{D-2} A_{h_i}$$

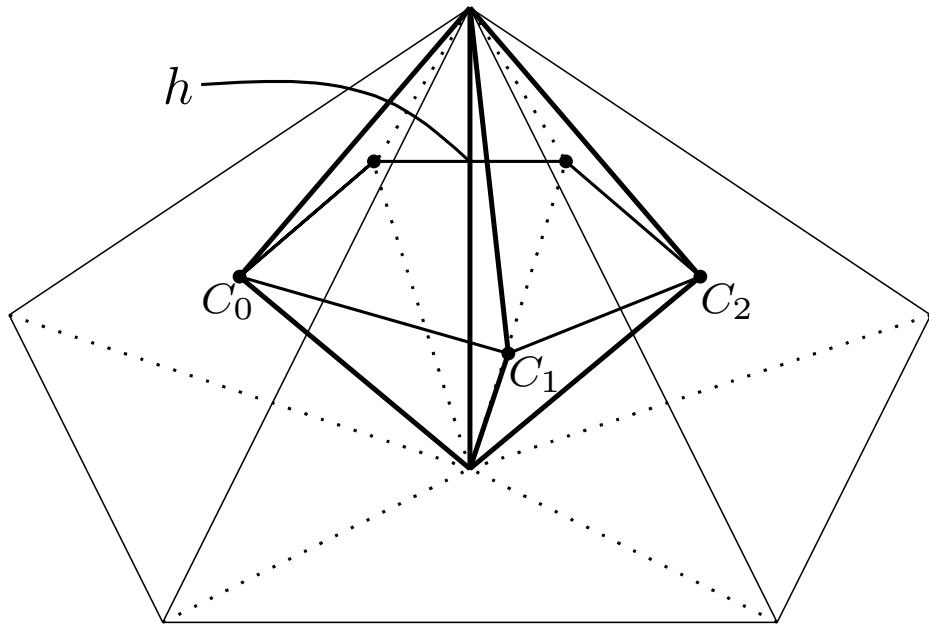
# Proper Volume Element の定義



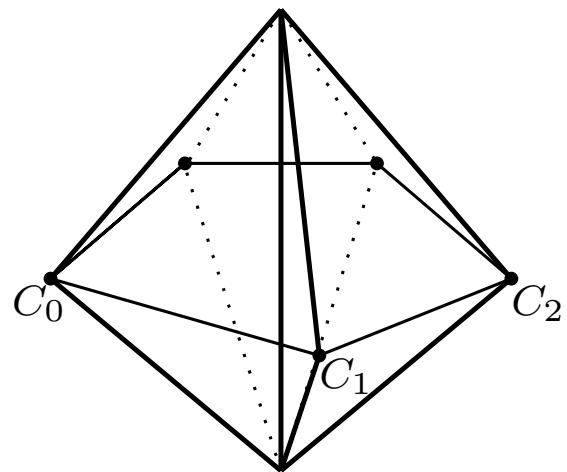
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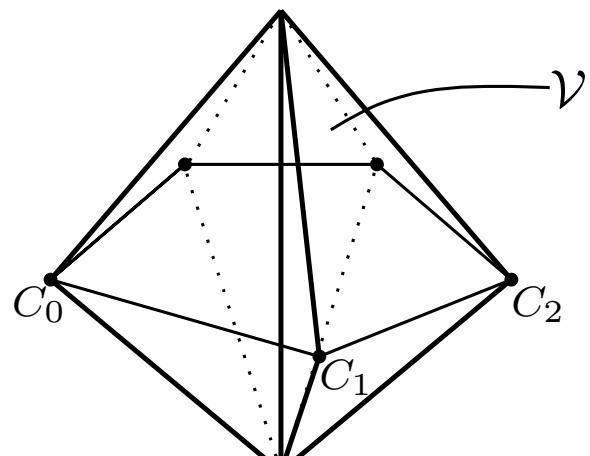
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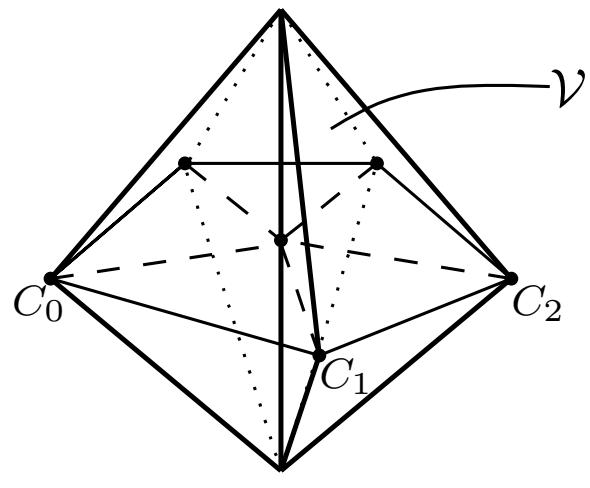


# Proper Volume Element の定義



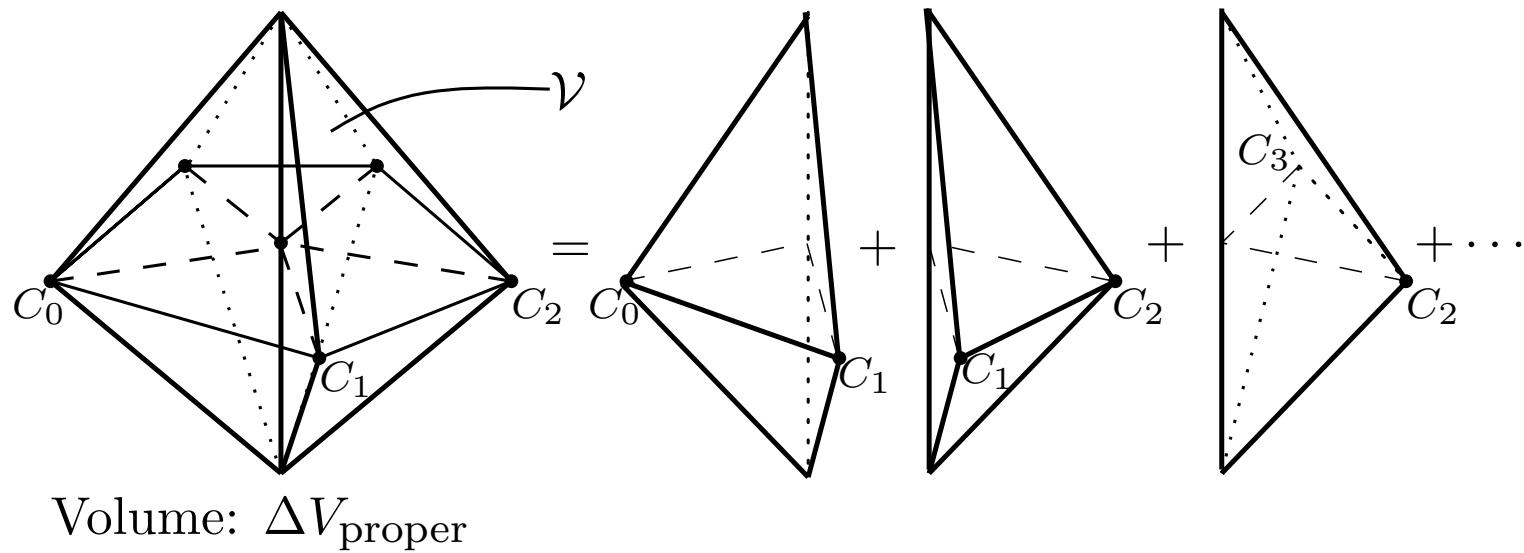
Volume:  $\Delta V_{\text{proper}}$

# Proper Volume Element の定義

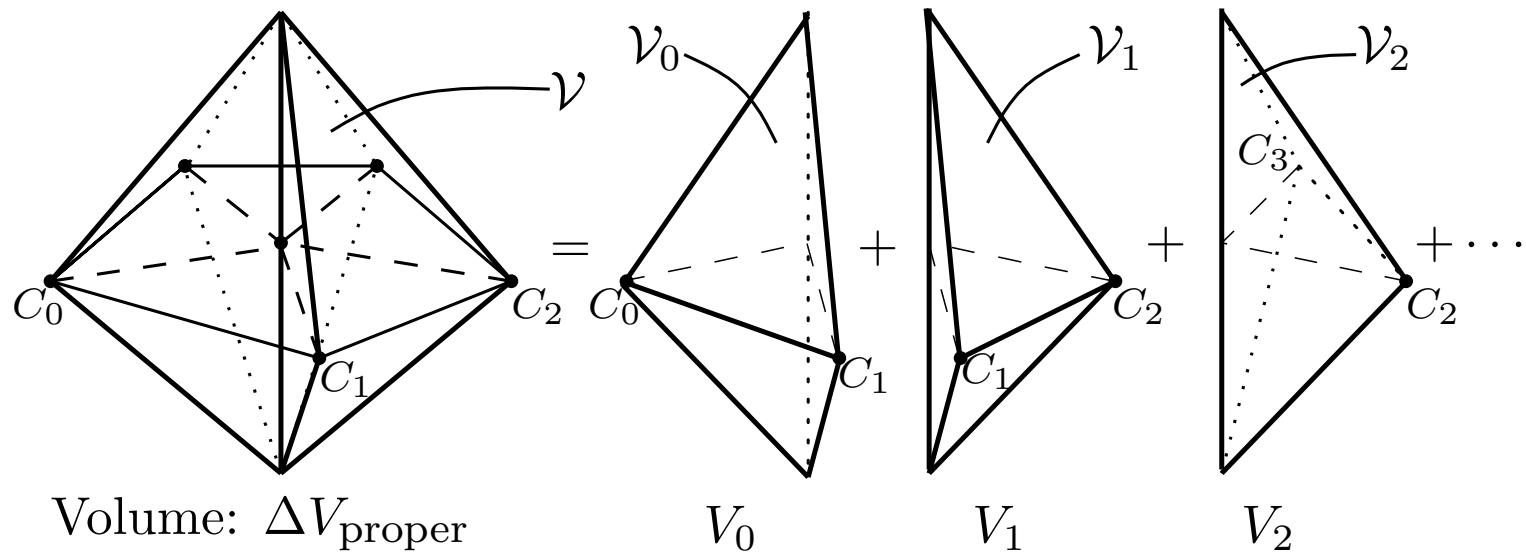


Volume:  $\Delta V_{\text{proper}}$

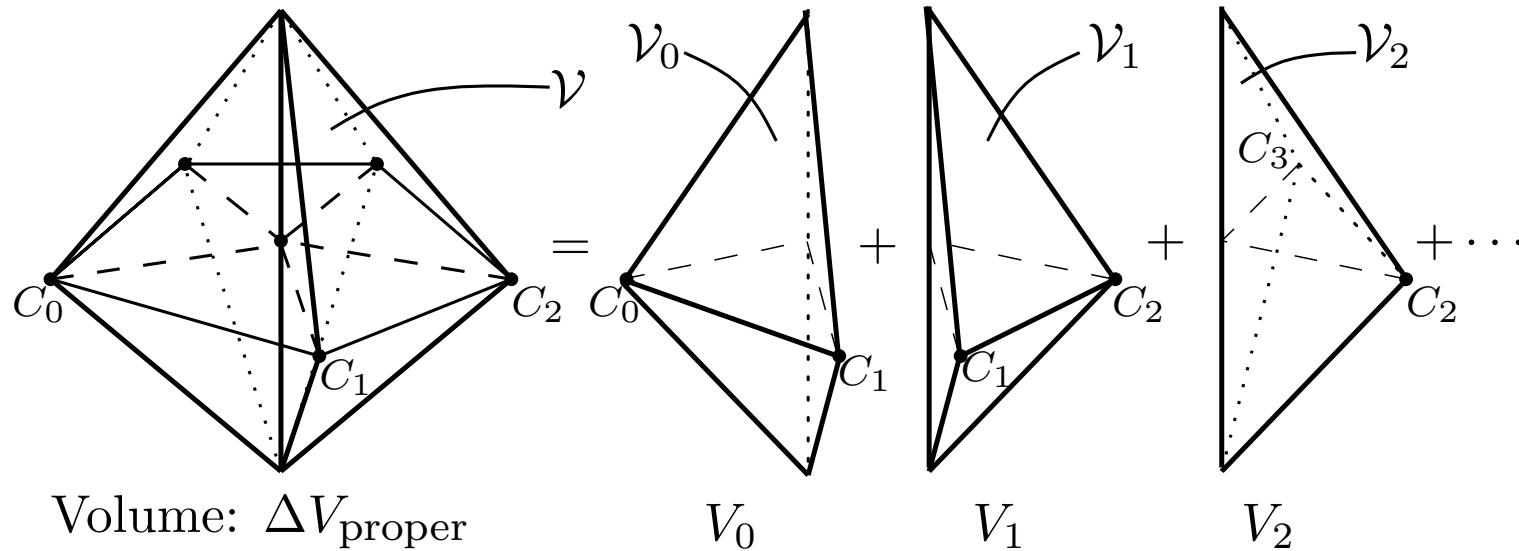
# Proper Volume Element の定義



# Proper Volume Element の定義



# Proper Volume Element の定義



$$\mathcal{V}_i = \{H_0, H_1, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h}\}$$

$$\Delta V_{\text{proper}} = \sum_{i=0}^{N_h-1} V_i$$

# $\mathcal{V}_i$ の分解

$$\mathcal{V}_i = \{ H_0, H_1, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h} \}$$

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# $\mathcal{V}_i$ の分解

$$\begin{aligned}\mathcal{V}_i &= \{ \underline{H_0, H_1, \dots, H_{D-2}}, C_i, C_{i+1 \bmod N_h} \} \\ &= h \rightarrow \sum_{j=0}^{D-2} h_j = \sum_{j=0}^{D-2} \{ O_h, H_0, \dots, H_{j-1}, \wedge, H_{j+1}, \dots, H_{D-2} \}\end{aligned}$$

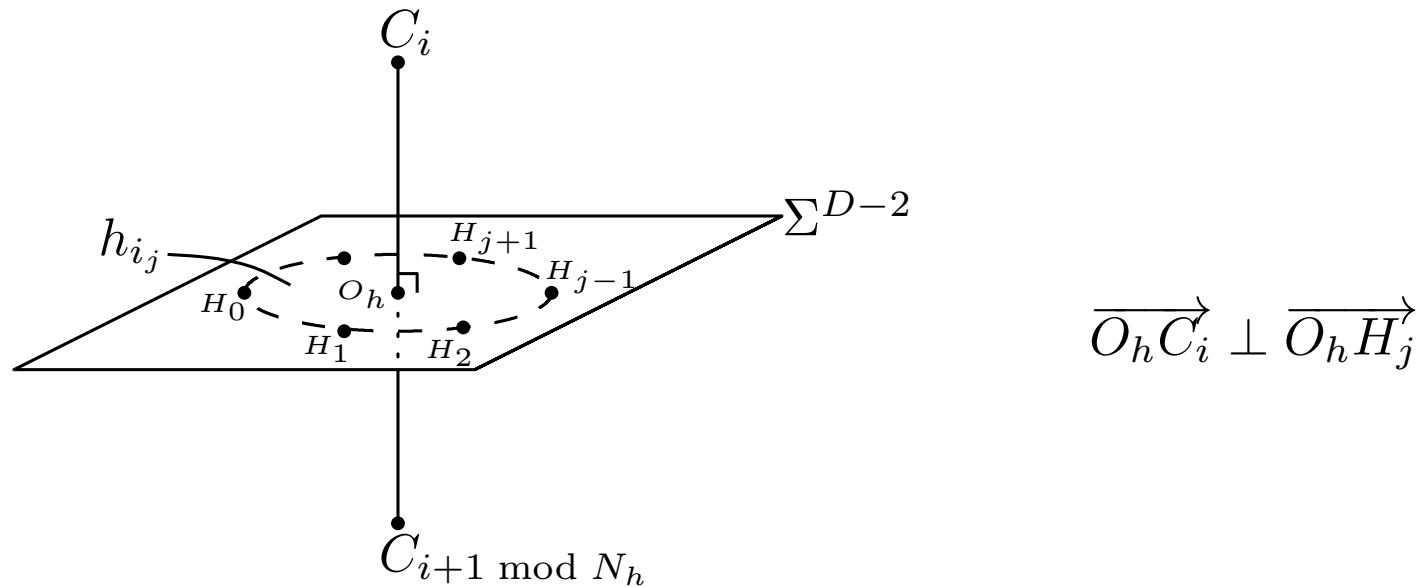
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$$\begin{aligned}\mathcal{V}_{i_j} &\equiv \{ O_h, H_0, \dots, H_{j-1}, \wedge, H_{j+1}, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h} \} \\ \text{Vol. } (\mathcal{V}_{i_j}) &= V_{i_j} & V_i &= \sum_{j=0}^{D-2} V_{i_j}\end{aligned}$$

# $\mathcal{V}_{i_j}$ の構造

$$\mathcal{V}_{i_j} \equiv \left\{ \frac{O_h, H_0, \dots, H_{j-1}, \wedge, H_{j+1}, \dots, H_{D-2}, C_i, C_{i+1 \bmod N_h}}{\text{頂点 } D+1 \text{ 個} \dots \dots \mathcal{V}_{i_j} \text{ は } D\text{-単体}} \right\}$$



# $V_{i_j}$ の値

$$\begin{aligned} V_{i_j} &= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\ &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\ &= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\ &\quad \times \left( \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \end{aligned}$$

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# $V_{i_j}$ の値

$$\begin{aligned} V_{i_j} &= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\ &\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\ &= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\ &\quad = (D-2)! A_{h_j} \\ &\quad \times \left( \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \end{aligned}$$

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&= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
&\quad = (D-2)! A_{h_j} \\
&\quad \times \left( \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
&\quad = 2 A_{h_i}^*
\end{aligned}$$

# $V_{i_j}$ の値

$$\begin{aligned}
V_{i_j} &= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right. \\
&\quad \left. \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
&= \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \cdots \wedge \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \cdots \wedge \overrightarrow{O_h H_{D-2}} \right) \\
&\quad = (D-2)! A_{h_j} \\
&\quad \times \left( \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right) \\
&\quad = 2 A_{h_i}^* \\
&= \frac{2}{D(D-1)} A_{h_j} A_{h_i}^*
\end{aligned}$$

# Proper Volume Element in Regge Calculus

$$\begin{aligned}\Delta V_{\text{proper}} &= \sum_{i=0}^{N_h-1} \sum_{j=0}^{D-2} V_{i_j} \\ &= \sum_{i=0}^{N_h-1} \sum_{j=0}^{D-2} \frac{2}{D(D-1)} A_{h_j} A_{h_i}^* \\ &= \frac{2}{D(D-1)} A_h A_h^*\end{aligned}$$

## 2. Regge Calculus における Hilbert 作用

(W. A. Miller, Class. Quantum Crav. **14**, 199 (1997))

- Ricci Scalar Curvature
- Proper Volume Element
- **Hilbert Action**

# Hilbert Action in Regge Calculus

$$\begin{aligned} S_{\text{Regge}} &= \frac{1}{16\pi} \sum_{\text{hinges}} \Delta V_{\text{proper}} R_{\text{Regge}} \\ &= \frac{1}{16\pi} \sum_{\text{hinges}} \left( \frac{2}{D(D-1)} A_h A_h^* \right) \left( D(D-1) \frac{\varepsilon_h}{A_h^*} \right) \\ &= \frac{1}{8\pi} \sum_{\text{hinges}} A_h \varepsilon_h \end{aligned}$$

### 3. Regge Calculus における Einstein 方程式

---

# Regge Analogue of Metric

計量テンソル … (大雑把に言うと) 時空の形を決めるもの

$n$ -単体 …  $\frac{n(n+1)}{2}$  本の辺の長さ全てを指定すると, 形が一意的に決定する.

※一般の立体ではこうはいかない

2次元の例 : 三角形 … 辺の長さを決めると形が決まる.

: 四角形 … 辺の長さが等しい菱形が存在する.

基本変数 :  $g_{\mu\nu} \longrightarrow l_p$  : Delaunay 格子に含まれる辺のうちの  $p$  番目のものの  
長さ

# Hilbert Action の変分

(L. Schläfli, Quart. J. Pure Appl. Math. **2**, 269 (1858))

$$\frac{\partial S_{\text{Regge}}}{\partial l_p} = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial}{\partial l_p} (A_h \varepsilon_h) = \frac{1}{8\pi} \left[ \sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h + \sum_{\text{hinges}} A_h \frac{\partial \varepsilon_h}{\partial l_p} \right] \\ = 0 \quad (\text{Schläfli Identity})$$

## 真空の Einstein 方程式

$$0 = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h$$

## 4. Expanding Polyhedral Universe

... Applying the Regge Calculus to  $(2 + 1)$ -dimensional closed FLRW universe

(P. A. Collins and R. M. Williams, Phys. Rev. **D7**, 965 (1973))

(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 024032 (2016))

(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 023502 (2016))

# 準備：任意格子への拡張

Delaunay 格子（単体格子）について Hilbert Action を組んだが、実は任意の格子に対して同様の形式が適用可能であることが分かっている。

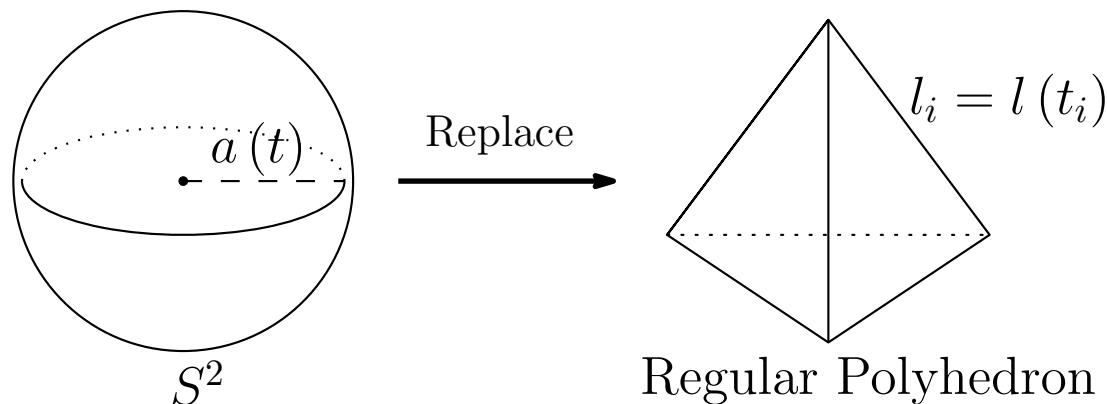
さらに宇宙項を入れると

$$S_{\text{Regge}} = \frac{1}{8\pi} \left( \sum_{i \in \{\text{hinges}\}} A_i \varepsilon_i - \Lambda \sum_{i \in \{\text{blocks}\}} V_i \right)$$

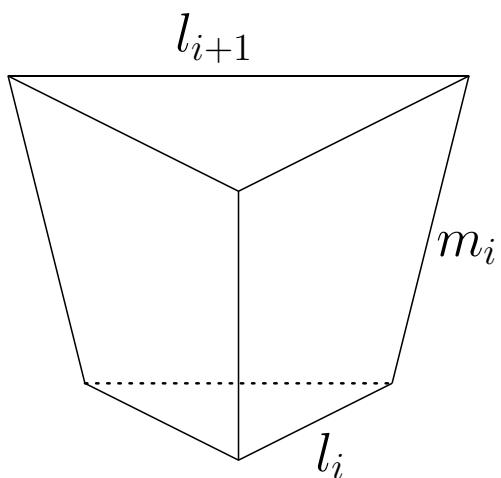
- $V_i$  : 任意の形の格子胞 (Block) の体積

# Regular Polyhedral Universe

球面状の Cauchy surface を正多面体で置き換える。

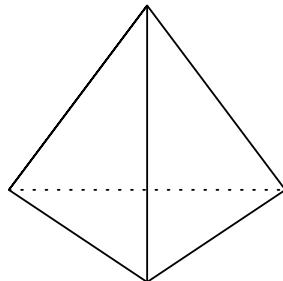


Block は正多角形を底面にもつ錐台

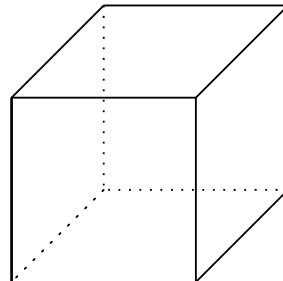


# Schl fli Symbols

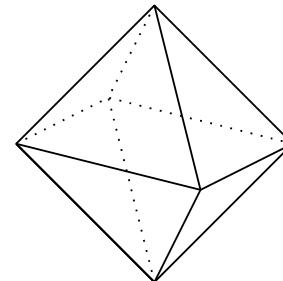
There are only five types of regular polyhedrons specified by the Schl fli symbols



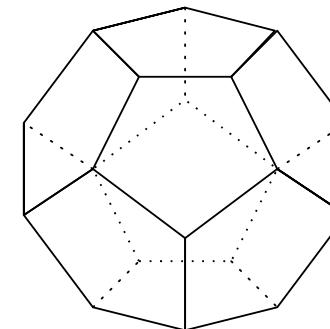
$\{3, 3\}$   
Tetrahedron



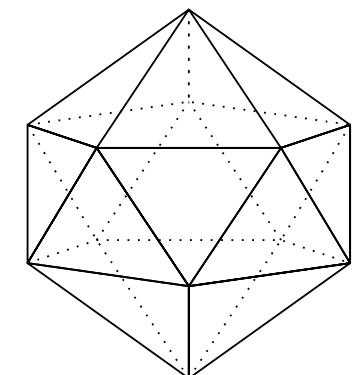
$\{4, 3\}$   
Cube



$\{3, 4\}$   
Octahedron



$\{5, 3\}$   
Dodecahedron



$\{3, 5\}$   
Icosahedron

Schl fli Symbols  $\{p, q\}$  ...

- $p$  : the number of the sides of a face
- $q$  : the number of faces around each vertex

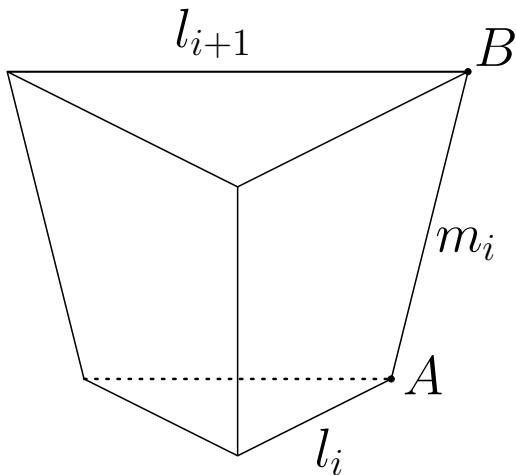
# Equations of motion

$$\begin{aligned}
 2\pi - q \arccos \left( -\frac{4m_i^2 \cos \frac{2\pi}{p} + \delta l_i^2}{4m_i^2 - \delta l_i^2} \right) &= \frac{q\Lambda}{12} \frac{(l_{i+1}^2 + l_{i+1}l_i + l_i^2)m_i}{\sqrt{m_i^2 - \frac{1}{4}\delta l_i^2 \csc^2 \frac{\pi}{p}}} \cot \frac{\pi}{p}, \\
 \arccos \frac{\delta l_i \cot \frac{\pi}{p}}{\sqrt{4m_i^2 - \delta l_i^2}} - \arccos \frac{\delta l_{i-1} \cot \frac{\pi}{p}}{\sqrt{4m_{i-1}^2 - \delta l_{i-1}^2}} &= \frac{\Lambda}{12} \left[ \frac{(l_{i+1} + 2l_i)m_i^2 + \frac{3}{4}l_i^2 \delta l_i \csc^2 \frac{\pi}{p}}{\sqrt{m_i^2 - \frac{1}{4}\delta l_i^2 \csc^2 \frac{\pi}{p}}} \right. \\
 &\quad \left. + \frac{(2l_i + l_{i-1})m_{i-1}^2 - \frac{3}{4}l_i^2 \delta l_{i-1} \csc^2 \frac{\pi}{p}}{\sqrt{m_{i-1}^2 - \frac{1}{4}\delta l_{i-1}^2 \csc^2 \frac{\pi}{p}}} \right] \cot \frac{\pi}{p}.
 \end{aligned}$$

To compare with the continuum theory, we introduce a circumradius of the polyhedron as an analogue of the scale factor

$$a_{Ri} = \frac{l_i \sin \frac{\pi}{q}}{2 \sqrt{\sin^2 \frac{\pi}{p} - \cos^2 \frac{\pi}{q}}}$$

# Strut length $m_i$ はなにか?



点  $A$  と点  $B$  は空間的には同一

→  $m_i$  は多面体の頂点に置かれた時計が示す経過固有時と考え

$$m_i = i\delta t_i$$

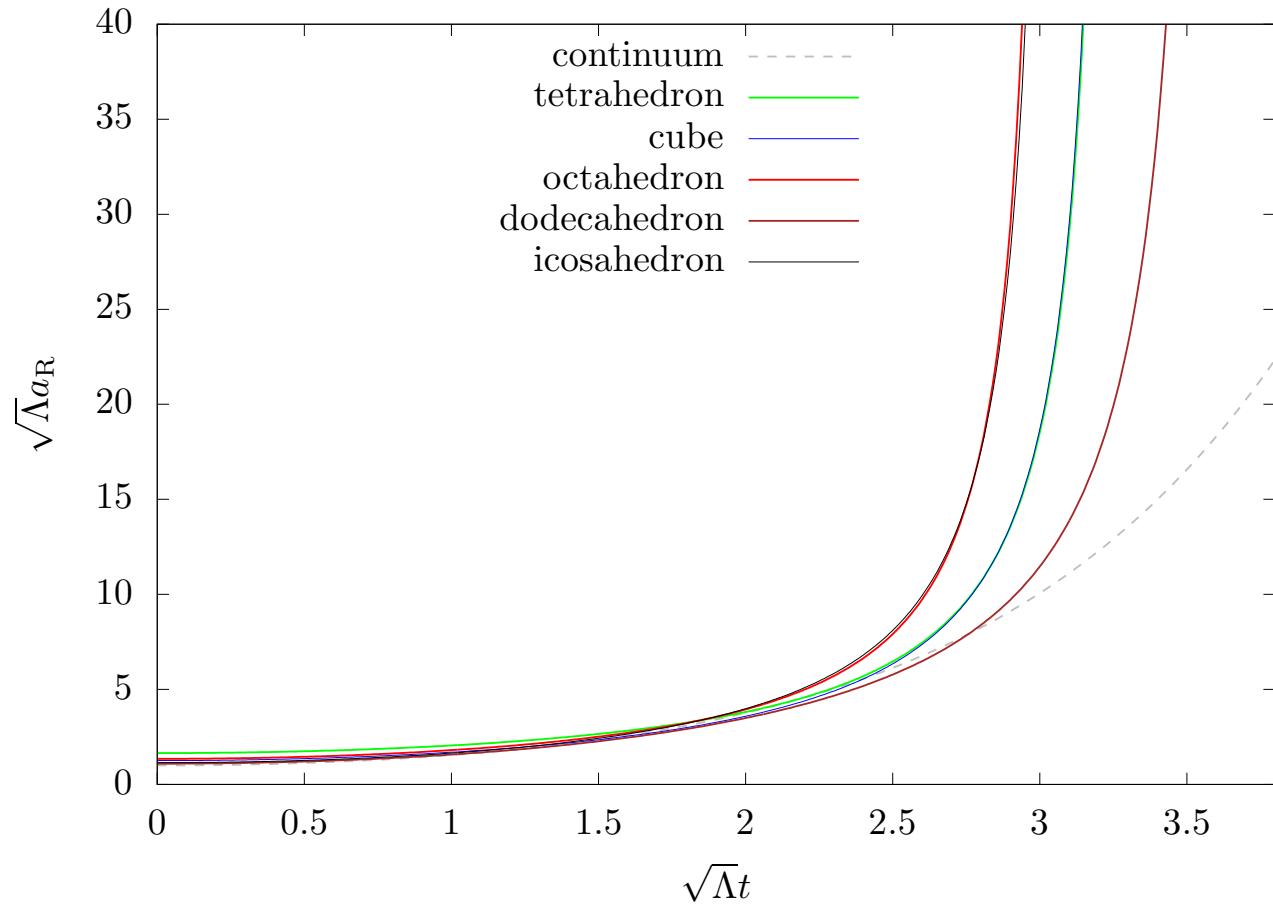
とする。

# Continuum time limit

$$m_i = i\delta t_i \rightarrow idt$$

$$\begin{aligned} 2\pi - q \arccos \left( \frac{\dot{l}^2 - 4 \cos \frac{2\pi}{p}}{4 + \dot{l}^2} \right) &= \frac{q\Lambda}{2} \frac{l^2 \cos \frac{\pi}{p}}{\sqrt{4 \sin^2 \frac{\pi}{p} + \dot{l}^2}}, \\ \frac{\ddot{l}}{4 + \dot{l}^2} &= \frac{\Lambda}{4} l \left[ 1 - \frac{l \ddot{l}}{2 \left( 4 \sin^2 \frac{\pi}{p} + \dot{l}^2 \right)} \right], \\ a_R(t) &= \frac{l(t) \sin \frac{\pi}{q}}{2 \sqrt{\sin^2 \frac{\pi}{p} \cos^2 \frac{\pi}{q}}} \end{aligned}$$

# Numerical Solution

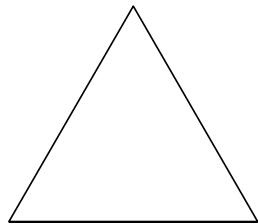


- finite-precision approximation of FLRW
- expands to infinite within finite time

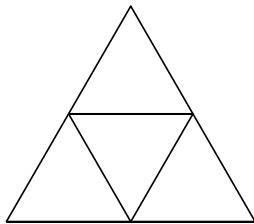
# Geodesic Domes

Frequency  $\nu \cdots$  Degree of subdivision

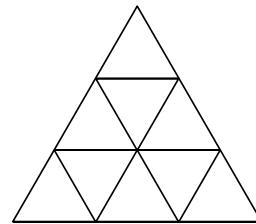
frequency 1



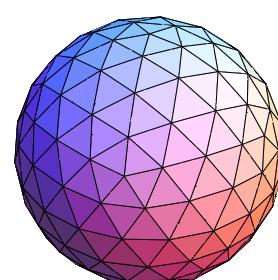
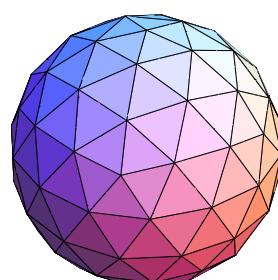
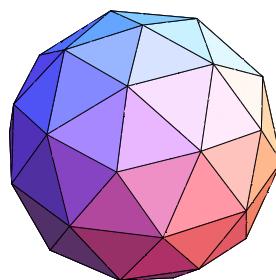
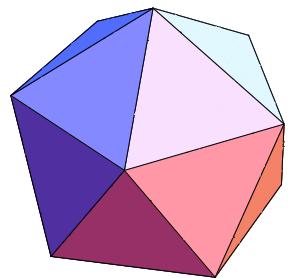
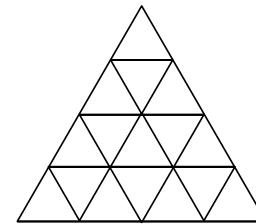
frequency 2



frequency 3



frequency 4



Better approximates a sphere than the regular polyhedrons.

# Equations of Motion for Geodesic Domes

$$\nu = 2$$

$$\frac{1}{5}\varepsilon_1 + \frac{1}{2}\varepsilon_2 = \Lambda l^2 \left[ \frac{\sin \xi \cos \frac{\xi}{2}}{\sqrt{4 \cos^2 \frac{\xi}{2} + \dot{l}^2}} + \frac{\sin^2 \frac{\xi}{2}}{\sqrt{3 + 4\dot{l}^2 \sin^2 \frac{\xi}{2}}} \right]$$

$$\xi = \text{const.}$$

$$\varepsilon_1 = 2\pi - 5\theta_{1,1}$$

$$\varepsilon_2 = 2\pi - 4\theta_{2,1} - 2\theta_{2,2}$$

$$\theta_{1,1} = \arccos \frac{4 \cos \xi + \dot{l}^2}{4 + \dot{l}^2}$$

$$\theta_{2,1} = \arccos \frac{\left(2 + \dot{l}^2\right) \sin \frac{\xi}{2}}{\sqrt{\left(4 + \dot{l}^2\right) \left(1 + \dot{l}^2 \sin^2 \frac{\xi}{2}\right)}}$$

$$\theta_{2,2} = \arccos \frac{1 + 2\dot{l}^2 \sin^2 \frac{\xi}{2}}{2 \left(1 + \dot{l}^2 \sin^2 \frac{\xi}{2}\right)}$$

# Equations of Motion for Geodesic Domes

$$\nu = 3$$

$$\frac{1}{5}\varepsilon_1 + \varepsilon_2 + \frac{1}{3}\varepsilon_3 = \Lambda l^2 \left[ \frac{\sin \xi \cos \frac{\xi}{2}}{\sqrt{4 \cos^2 \frac{\xi}{2} + i^2}} + \frac{2 \sin^2 \frac{\xi}{2} \cos^2 \frac{\eta}{2}}{\sqrt{\sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}} + \frac{\sin^2 \frac{\xi}{2} \csc \frac{\eta}{2} \sin \zeta \cos \zeta}{\sqrt{4 \sin^2 \frac{\eta}{2} \cos^2 \frac{\zeta}{2} + i^2 \sin^2 \frac{\xi}{2}}} \right]$$

$$\xi = \text{const.}$$

$$\eta = \text{const.}$$

$$\zeta = \text{const.}$$

$$\varepsilon_1 = 2\pi - 5\theta_{1,1}$$

$$\varepsilon_2 = 2\pi - 4\theta_{2,1} - 2(\theta_{2,1} + \theta_{2,2} + \theta_{2,3})$$

$$\varepsilon_3 = 3(\theta_{3,2} + \theta_{3,3})$$

$$\theta_{1,1} = \arccos \frac{4 \cos \xi + i^2}{4 + i^2} \quad , \quad \theta_{2,1} = \arccos \frac{(2 + i^2) \sin \frac{\xi}{2}}{\sqrt{(4 + i^2)(1 + i^2 \sin^2 \frac{\xi}{2})}}$$

$$\theta_{2,2} = \arccos \frac{2 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{\sqrt{(1 + i^2 \sin^2 \frac{\xi}{2})(4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2})}}, \quad \theta_{2,3} = \arccos \frac{(2 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}) \sin \frac{\xi}{2}}{\sqrt{(4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2})(\sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2} \sin^2 \frac{\zeta}{2})}}$$

$$\theta_{3,2} = \arccos \frac{4 \cos \eta \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}} \quad , \quad \theta_{3,3} = \arccos \frac{4 \cos \zeta \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}{4 \sin^2 \frac{\eta}{2} + i^2 \sin^2 \frac{\xi}{2}}$$

# Shortcomings of Geodesic Domes

- Frequency を上げると球面への近似精度は向上するが, 考慮すべきパラメータの数が指数関数的に増加する.
- $\nu \rightarrow \infty$  の極限では無限個のパラメータが必要になる.  
⇒ 解析的に連続極限を求めることが不可能.

# Pseudo-Regular Polyhedron Universe

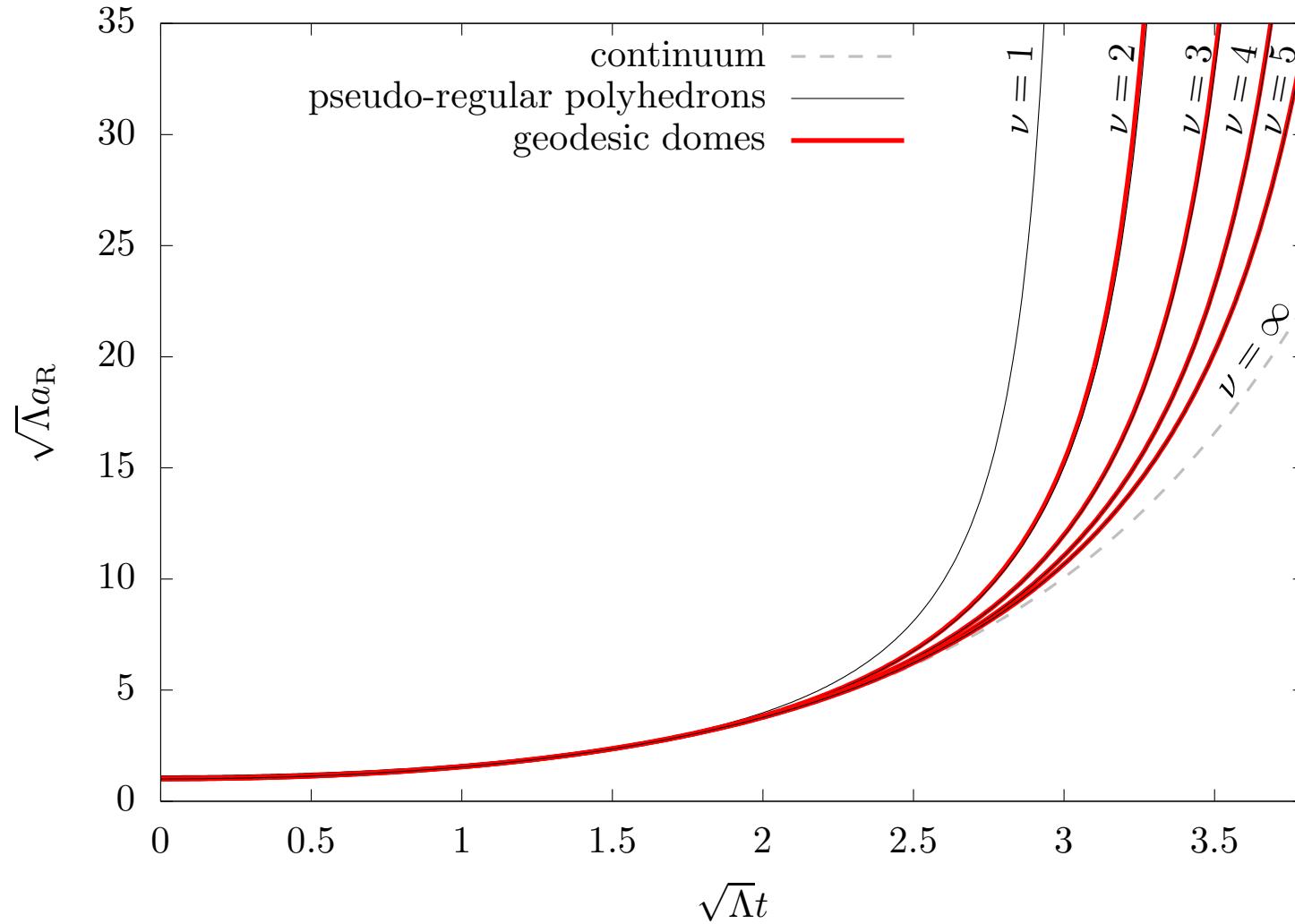
Regarding the geodesic domes as pseudo-regular polyhedrons, we may define the fractional Schläfli Symbol

$$\{p, q\} = \left\{ 3, \frac{30\nu^2}{5\nu^2 + 1} \right\}$$

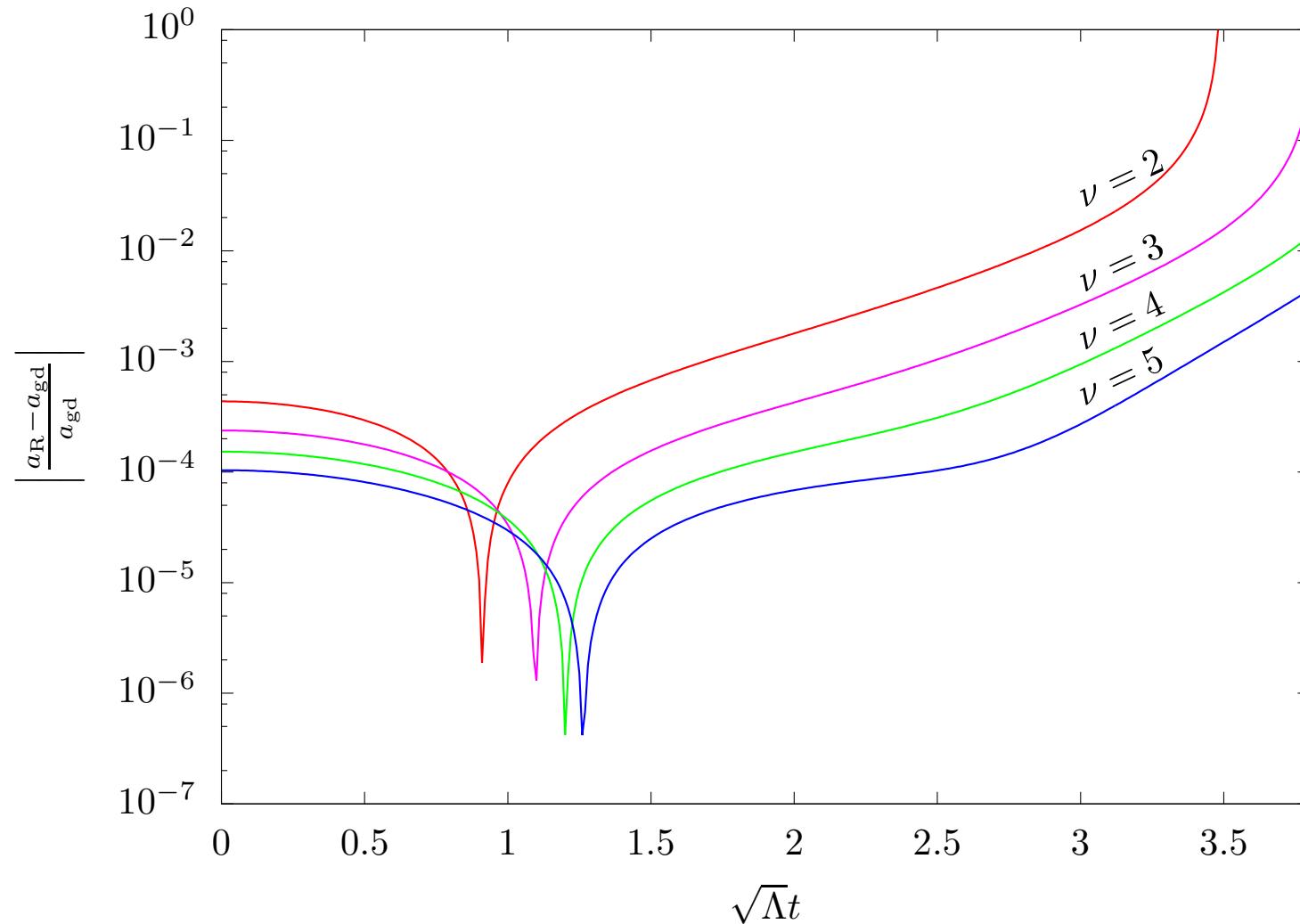
Equation of motion

$$\begin{aligned}\frac{\ddot{l}}{4 + \dot{l}^2} &= \frac{\Lambda}{4}l \left[ 1 - \frac{l\ddot{l}}{2(3 + l^2)} \right], \\ a_R(t) &= \frac{l(t)}{\sqrt{3}} \tan \frac{(5\nu^2 + 1)\pi}{30\nu^2}\end{aligned}$$

# Plots of Scale Factors



Deviation  $\left| \frac{a_R - a_{gd}}{a_{gd}} \right|$



# Infinite Frequency Limit

Taking the limit  $\nu \rightarrow \infty$

$$\left\{ \begin{array}{l} \frac{\ddot{l}}{4+l^2} = \frac{\Lambda}{4}l \left[ 1 - \frac{l\ddot{l}}{2(3+l^2)} \right] \\ a_R(t) = \frac{l(t)}{\sqrt{3}} \tan \frac{(5\nu^2+1)\pi}{30\nu^2} \end{array} \right.$$

$$\Rightarrow \ddot{a}_R = \Lambda a_R$$

Exactly coincides with Einstein equations in General relativity.

## 5. Oscillating 4-Polytopal Universe

... Applying the Regge Calculus to  $(3 + 1)$ -dimensional closed FLRW universe

(P. A. Collins and R. M. Williams, Phys. Rev. **D7**, 965 (1973))

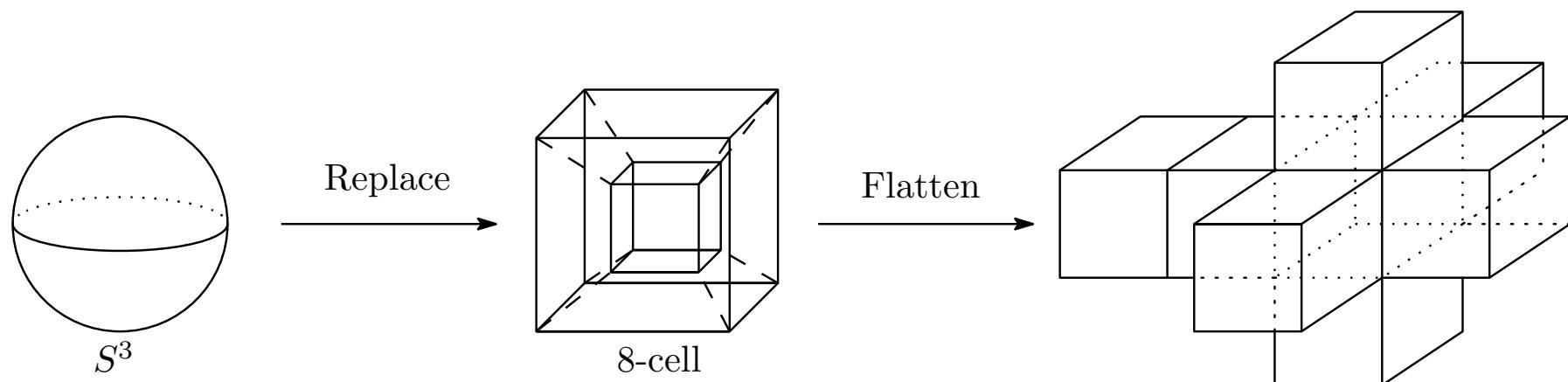
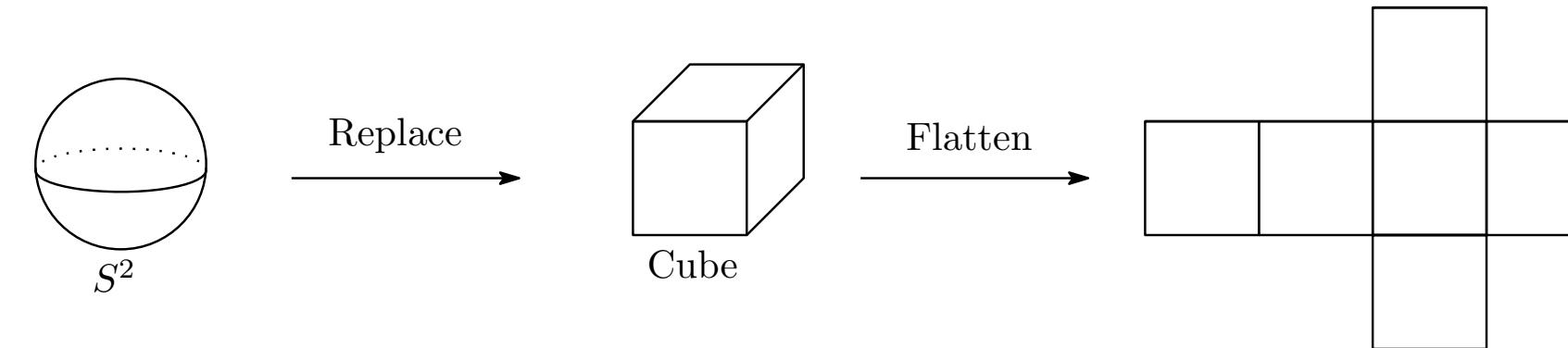
(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 024032 (2016))

(R. G. Liu and R. M. Williams, Phys. Rev. **D93**, 023502 (2016))

# 正多胞体による Cauchy surface $S^3$ の置き換え

(P. A. Collins and R. M. Williams, *Phys. Rev. D7*, 1973)

e.g.) 正八胞体による置き換え



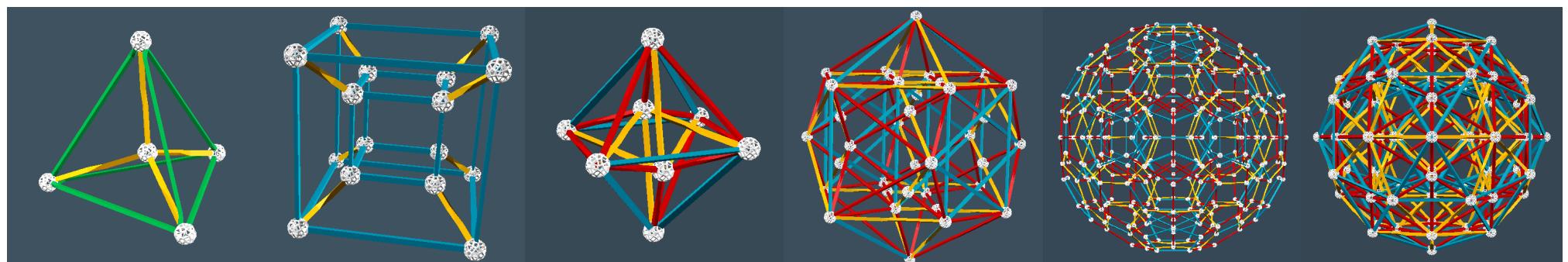
# 正多胞体と Schläfli 記号

	5-cell	8-cell	16-cell	24-cell	120-cell	600-cell
Schläfli 記号 $\{p, q, r\}$	$\{3, 3, 3\}$	$\{4, 3, 3\}$	$\{3, 3, 4\}$	$\{3, 4, 3\}$	$\{5, 3, 3\}$	$\{3, 3, 5\}$

$p$  : the number of sides of a face

$q$  : the number of faces around a vertex in a cell

$r$  : the number of cells around a edge



5-cell

8-cell

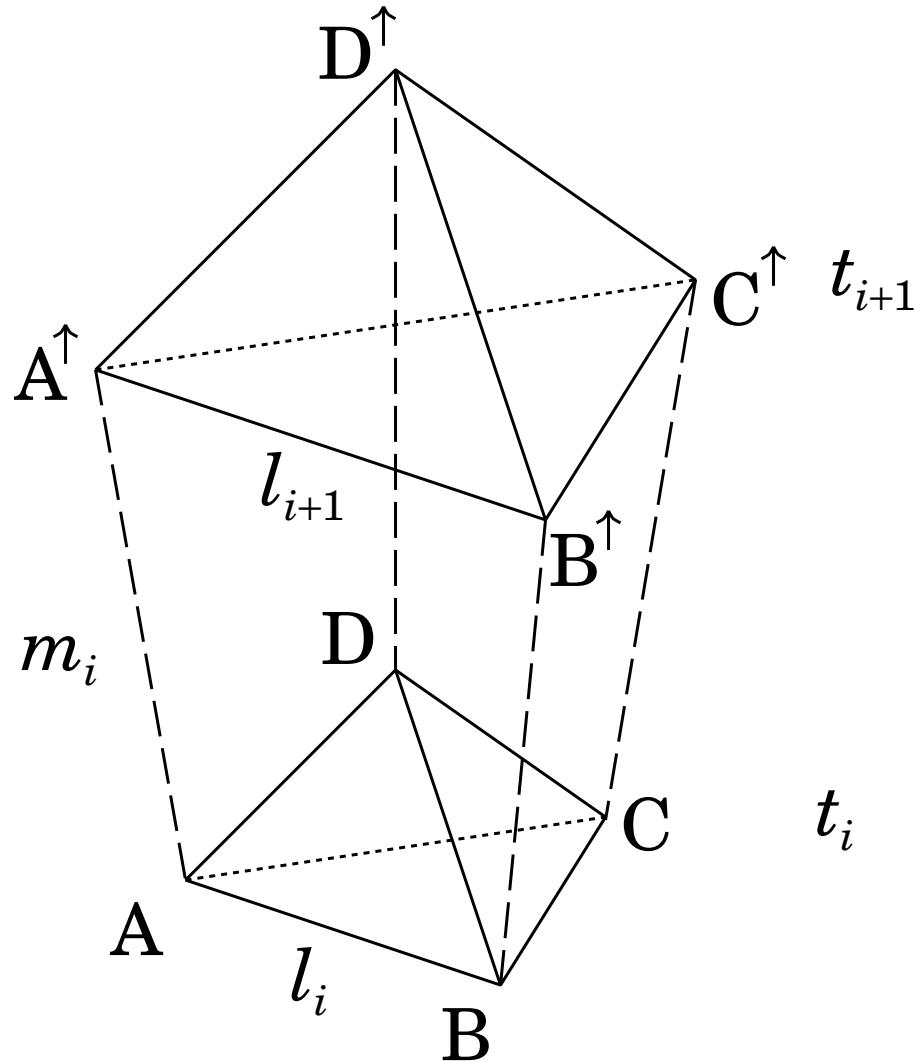
16-cell

24-cell

120-cell

600-cell

# Block



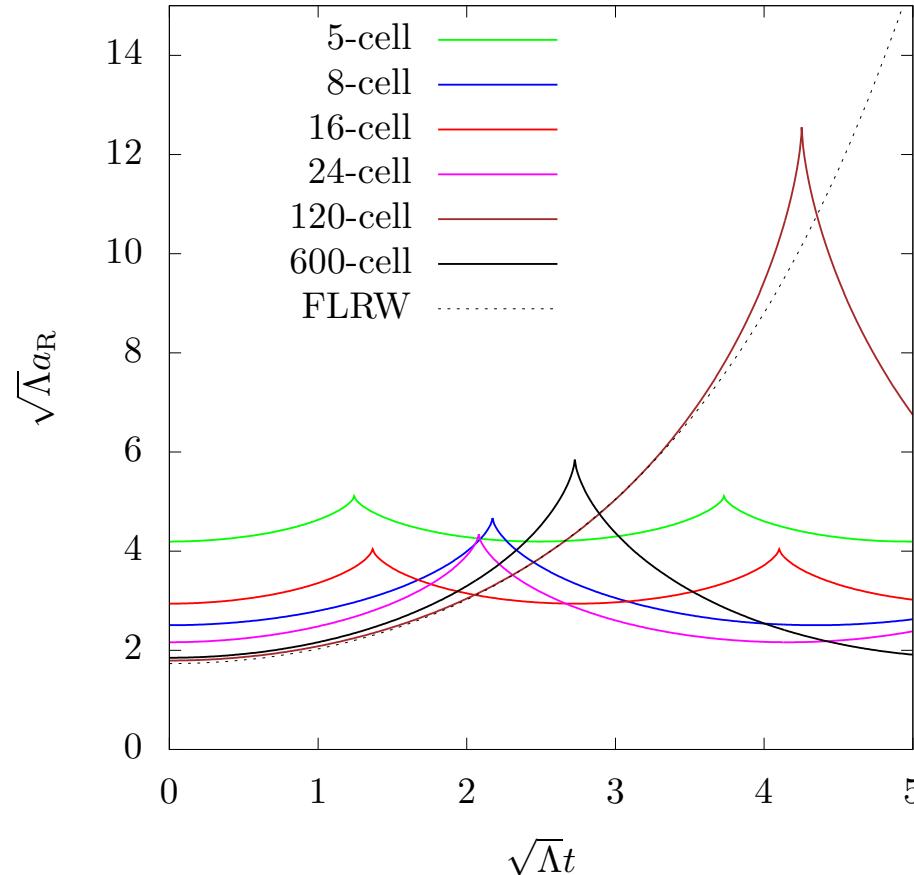
# Equations of motion with the continuum Lorentzian time

$$\begin{aligned}
& 2\pi - r \arccos \frac{4 \left( \sin^2 \frac{\pi}{p} - 2 \cos^2 \frac{\pi}{q} \right) - \dot{l}^2 \cos \frac{2\pi}{q}}{4 \sin^2 \frac{\pi}{p} + \dot{l}^2} \\
&= \frac{r\Lambda}{12} l^2 \sqrt{\frac{4 + \dot{l}^2}{4 \left( \sin^2 \frac{\pi}{p} - \cos^2 \frac{\pi}{q} \right) + \dot{l}^2 \sin^2 \frac{\pi}{q}}} \cot^2 \frac{\pi}{p} \cos \frac{\pi}{q}, \\
\ddot{l} &= \frac{\Lambda}{3} l \left( 1 + \frac{\dot{l}^2}{4 \sin^2 \frac{\pi}{p}} \right) \left[ 1 + \frac{1}{4} \dot{l}^2 - \frac{1}{2} \frac{l \ddot{l} \cos^2 \frac{\pi}{p}}{4 \left( \sin^2 \frac{\pi}{p} - \cos^2 \frac{\pi}{q} \right) + \dot{l}^2 \sin^2 \frac{\pi}{q}} \right].
\end{aligned}$$

Scale factor

$$a_R(t) = \frac{l(t)}{2} \sqrt{\frac{\sin^2 \frac{\pi}{q} - \cos^2 \frac{\pi}{r}}{\sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{r} - \cos^2 \frac{\pi}{q}}}$$

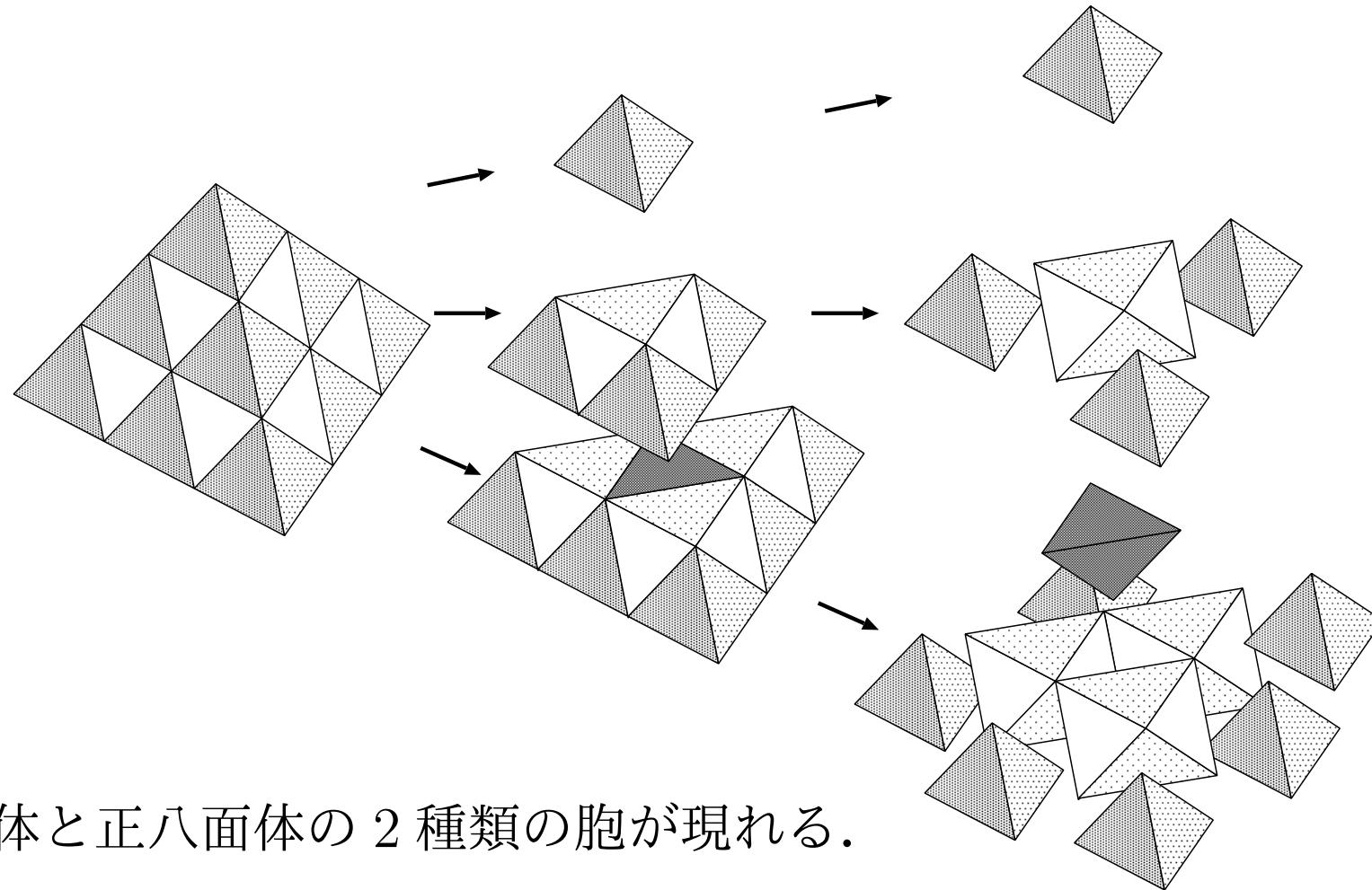
# Numerical Solution



- ・有限の時間, 有限の大きさで膨張速度が発散.
- ・その後収縮に転じ, 以降振動を繰り返す.

# Pseudo-Regular 4-Polytope Universe

600-cell の 3 次元胞（正四面体）の分割.



正四面体と正八面体の 2 種類の胞が現れる.

# Pseudo-Regular 4-Polytope Universe

胞が 2 種類ある場合、「Schläfli symbol の平均」による pseudo-regular 4-polytope の定義は機能しない。 $(\nu \rightarrow \infty$  で 3 次元球面を再現しない。)

$$p(\vartheta_2) = \frac{\pi}{\arccos(\sin \frac{\vartheta_2}{2})},$$

$$q(\vartheta_2, \vartheta_3) = \frac{\pi}{\arccos(\cos \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2})},$$

$$r(\vartheta_3, \vartheta_4) = \frac{\pi}{\arccos(\cos \frac{\vartheta_3}{2} \sin \frac{\vartheta_4}{2})}$$

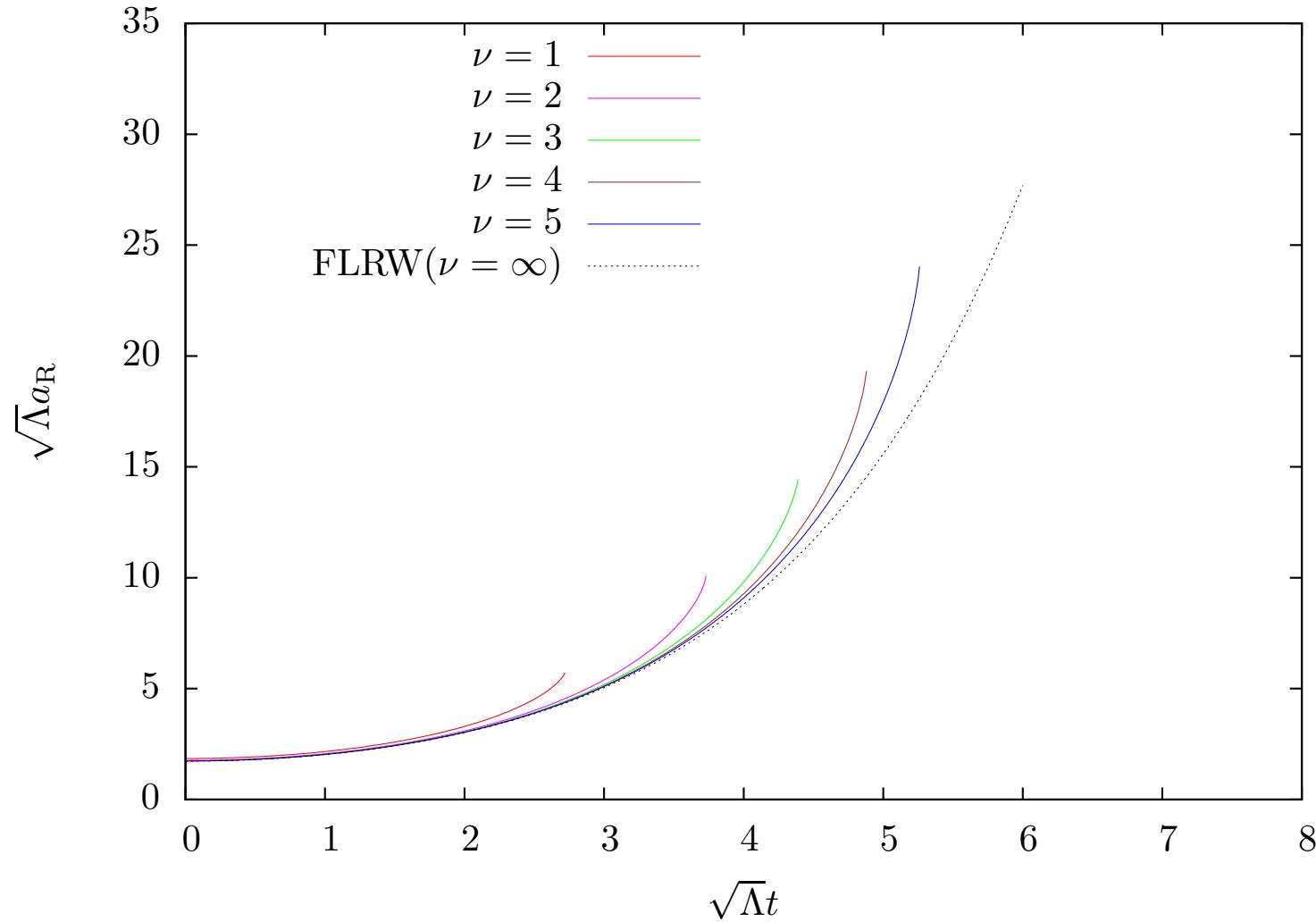
超二面角の平均をもって定義してやる。

正多角形の内角 :  $\vartheta_2 = \frac{\pi}{3}$ ,

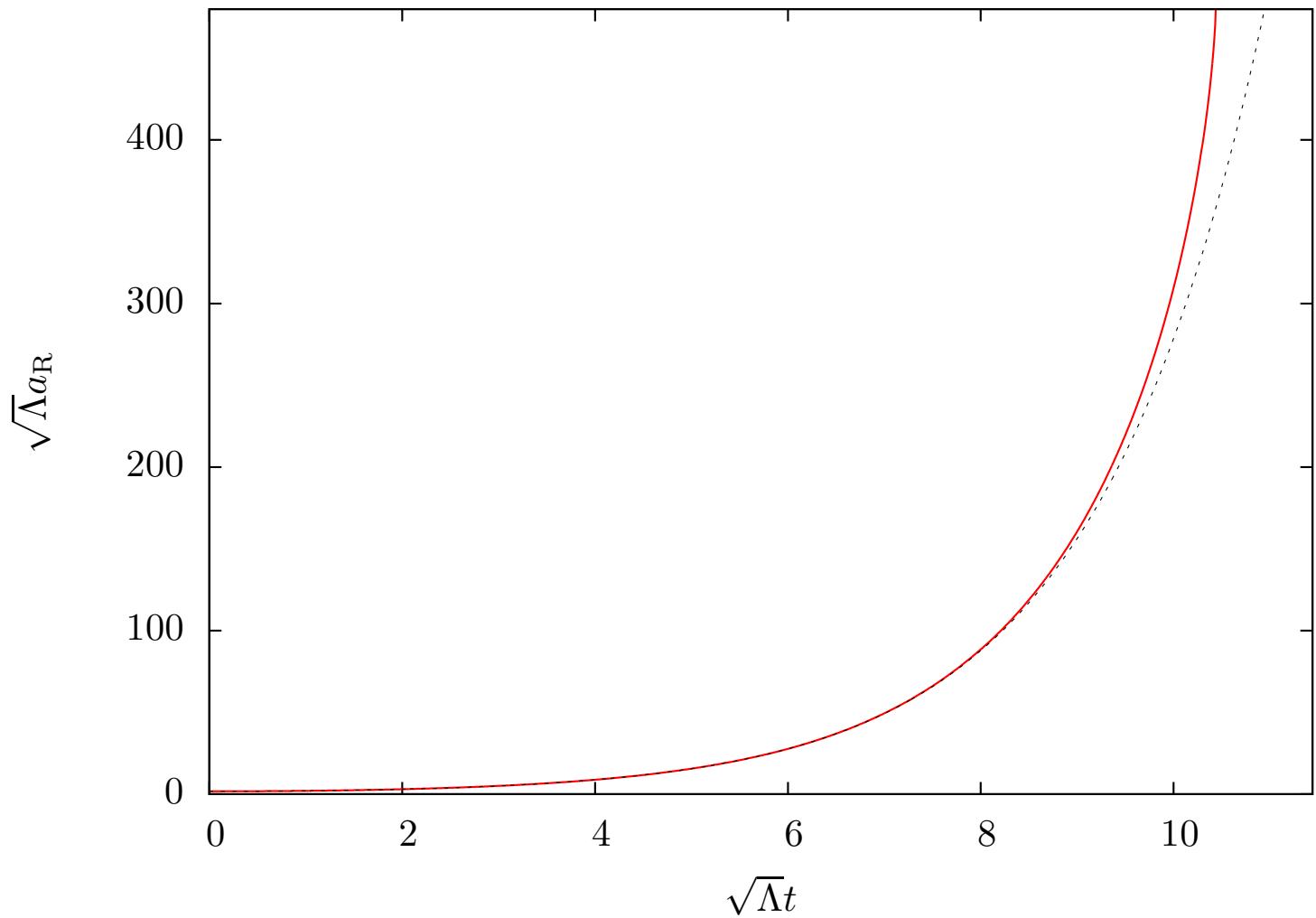
正多面体の二面角 :  $\vartheta_3 = \frac{(\nu^2 - 1)\pi + 3\arccos \frac{1}{3}}{2\nu^2 + 1}$ ,

正多胞体の二胞角 :  $\vartheta_4 = \frac{(2\nu^2 - 3\nu + 1)\pi + 3\nu \arccos \left(-\frac{3\sqrt{5}+1}{8}\right)}{2\nu^2 + 1}$

# Plots of Scale Factors



Large frequency:  $\nu = 100$



## 6. Summary and Future work

# Summary

## Regge Calculus

- 単体分割による計算の手法の一つ.
- 各単体の形状を変化させることで多様体の Dynamics を表現.

## Hilbert 作用

$$S_{\text{Regge}} = \frac{1}{16\pi} \sum_{\text{hinges}} \Delta V_{\text{proper}} R_{\text{Regge}} = \frac{1}{8\pi} \sum_{\text{hinges}} A_h \varepsilon_h$$

$A_h$  : Hinge Volume

$\varepsilon_h$  : Deficit Angle

## 真空の Einstein 方程式

$$0 = \frac{1}{8\pi} \sum_{\text{hinges}} \frac{\partial A_h}{\partial l_p} \varepsilon_h$$

# Summary

## Polyhedral Universe

- Geodesic Dome Model
  - 任意の精度で球面を近似.
  - Frequency を上げるごとに方程式が複雑化.
  - 連續極限への解析的アプローチ不可能.
- Pseudo-Regular Polyhedron Model
  - Geodesic Dome を非常に高精度で近似.
  - Frequency によらず方程式が单一かつ簡潔.
  - 連續極限を解析的に求めることが可能.

# Summary

## 4-Polytope Universe

- Regular 4-Polytope Model
  - 有限の周期と振幅で振動を繰り返す.
- Pseudo-Regular 4-Polytope Model
  - Frequency によらず方程式が单一かつ簡潔.
  - 連続極限を解析的に求めることが可能.

# Future work: $D$ -polytopal Universe

$$\Lambda_D = \frac{2\Lambda}{(D-1)(D-2)}.$$

